



NEW YORK UNIVERSITY
INSTITUTE OF MATHEMATICAL SCIENCES
LIBRARY

25 Washington Place, New York City

NEW YORK UNIVERSITY

Institute of Mathematical Sciences

Division of Electromagnetic Research

RESEARCH REPORT No. HSN-2

Analytical Theory of Turbulent Diffusion

P. H. ROBERTS

Contract No. N(0nr)-285(33)

JUNE, 1960

HSN-2
c.2

NEW YORK UNIVERSITY

Institute of Mathematical Sciences

Division of Electromagnetic Research

Research Report No. HSN-2

Analytical Theory of Turbulent Diffusion

P. H. ROBERTS [†]

P. H. Roberts
P.H. Roberts

This work was supported by the
Office of Naval Research under
Contract No. Nonr-285(33)

Robert H. Kraichnan
Robert H. Kraichnan
Principal Investigator

[†] Permanent address:
Dept. of Physics, King's College,
Newcastle-upon-Tyne, England

Morris Kline
Morris Kline
Director

June, 1960

Abstract

Recently Kraichnan (1959) has propounded a theory of homogeneous turbulence, based on a novel perturbation method, that leads to closed equations for the velocity covariance. In the present paper, this method is applied to the theory of turbulent diffusion and closed equations are derived for the probability distributions of the positions of marked fluid elements released in a turbulent flow.

Two topics are discussed in detail. The first is the probability distribution, at time t , of the displacement of an element from its initial position. In homogeneous flows, this distribution is found to resemble that for classical diffusion but with a variable coefficient of diffusion which is proportional to $v_o^2 t$ for $t \ll \ell/v_o$ and which approaches a constant value $\doteq \ell v_o$ for $t \gg \ell/v_o$ (ℓ = macroscale, v_o = r.m.s. turbulent velocity).

The second topic treated is the joint probability distribution of the displacements of two fluid elements. Particular attention is focused upon the probability distribution of relative displacement, i.e. Richardson's distance-neighbour function. This is found to be Gaussian for separations r which are large ($\gg \ell$). For smaller separations ($r \ll \ell$), its behavior at high Reynolds numbers is found to be quite well expressed in terms of a variable diffusion coefficient $K(r,t)$, as suggested by Richardson (1926). For all but extremely short times, $K(r,t)$ is found to depend only on r and on the form of the inertial range spectrum $E(k)$.

On assuming $E(k) \propto v_o^2 \ell (k\ell)^{-3/2}$ as results from Kraichnan's approximation (1959), one finds $K(r) \propto v_o \ell (r/\ell)^{3/2}$. On the basis of similarity

arguments of the Kolmogorov type, which give $E(k) \propto v_0^2 \ell(k\ell)^{-5/3}$, one finds $K(r) \propto v_0 \ell(r/\ell)^{4/3}$ as, in fact, Richardson originally proposed. The dispersion $\langle r^2 \rangle$ is proportional to $\ell^2(v_0 t/\ell)^4$ on Kraichnan's theory; while $\langle r^2 \rangle \propto \ell^2(v_0 t/\ell)^3$ on the similarity theory. This illustrates that the behavior of $\langle r^2 \rangle$ is very sensitive to the spectrum.

Table of Contents

	Page
1. Introduction	1
2. Diffusion from a fixed source	2
2.1 Methods of approach	2
2.2 Exact results for short times	4
2.3 An integro-differential equation for $G(x,t x_0, t_0)$	8
2.4 Solution of the integro-differential equation for short times	12
2.5 Solution for large times	15
3. Relative diffusion	19
3.1 Formulation of problem: exact results for short times	19
3.2 Integral equations for $G(x,t;y,s x_0, t_0; y_0, s_0)$	24
3.3 Solution of equations for short times	28
3.4 Solutions for large and intermediate times	32
Appendix A: Derivation of equation (2.26)	44
Appendix B: Derivation of equation (3.18)	56
References	64

1. INTRODUCTION

The aim of the theory of turbulent diffusion is to determine in a statistical sense the migration of marked particles as they are carried along with a turbulent flow. Like molecular motion in a dilute gas of discrete particles, turbulent diffusion is a linear process if the convected particles have no reaction on the flow; i.e. the probability distribution for the position of a marked particle in space obeys the superposition rule and changes in time according to a linear equation. Unlike classical molecular motion, the motion of neighboring fluid elements in a continuum is correlated, although one expects that over distances large compared to the macroscale ℓ of the turbulence this correlation is weak and that elements separated by such distances move almost independently. Furthermore, unlike classical molecular diffusion, turbulent diffusion is not a Markoff process. However, one expects that over times large compared to ℓ/v_0 , where v_0 is the root mean square fluid velocity, the fluid elements will suffer many essentially uncorrelated deflections by the energy containing eddies and that accordingly their motion over such long times will be almost Brownian. Under such circumstances, one expects that the spread of marked particles carried by the fluid will indeed resemble classical diffusion, and that it will be possible to define a coefficient of eddy diffusivity. An analytical basis for these qualitative observations is given in Sec. 2.

Particles which start out simultaneously at nearby points have closely similar histories in any one realization of the turbulent flow (and over times which are not too long). For such times, their relative motion is unaffected by eddies whose spatial scale is large compared to

the initial separation; such eddies give nearly equal displacements to the two points. The change in the separation r is governed by the smaller eddies, particularly those whose length scale is of the same order as r . Thus, in a flow of high Reynolds number, we expect that while the particles are separated by a distance appropriately small compared to ℓ their relative diffusion will be governed by the inertial range spectrum of the turbulent flow, and will be unaffected by the structure of the energy containing eddies. An analytical basis for these surmises is given in Sec. 3, and a form is proposed for the variable diffusion coefficient introduced by Richardson (1926).

2. DIFFUSION FROM A FIXED SOURCE

2.1 Methods of approach

There are two main ways of attempting to give an analytical framework to the qualitative arguments of Sec. 1. In the Lagrangian framework (as distinct from the Eulerian approach to be described presently), probability distributions are defined for the displacements, velocities, etc., of given marked particles, and the relationships between them are studied. For example, let $G(\underline{x}, t | \underline{x}_o, t_o) d\underline{x}$ be the probability that a fluid particle lying at the point \underline{x}_o at time t_o should, at the later time t , lie within a volume $d\underline{x}$ at the point \underline{x} . Let $V(\underline{x}, \underline{u}, t | \underline{x}_o, t_o) d\underline{x} du$ be the probability that this same particle should at that time lie within $d\underline{x}$ and have a velocity between \underline{u} and $\underline{u} + d\underline{u}$. Then it is not difficult to show that

$$\frac{\partial}{\partial t} G(\underline{x}, t | \underline{x}_o, t_o) + \frac{\partial}{\partial \underline{x}_i} \int V(\underline{x}, \underline{u}, t | \underline{x}_o, t_o) u_i du = 0. \quad (2.1)$$

(In 2.1 and elsewhere, we use the summation convention.)

Through the hydrodynamical equations, it is possible to derive a similar, though far more involved equation, relating V to another probability function, and this, in its turn, to yet another. It would be necessary to close this hierarchy of equations in some way in order to obtain from it an evaluation of $G(\underline{x}, t | \underline{x}_0, t_0)$.

The second approach (advocated by G.K. Batchelor, 1952b; see also W.H. Reid, 1955; P.H. Roberts, 1957) involves reformulating the problem of finding Lagrangian probability functions as one of determining certain Eulerian moments. A passive scalar quantity $\psi(\underline{x}, t)$ is introduced which satisfies the equation

$$\frac{\partial}{\partial t} \psi(\underline{x}, t) + \frac{\partial}{\partial \underline{x}_i} [\psi(\underline{x}, t) u_i(\underline{x}, t)] = 0, \quad (2.2)$$

where $u_i(\underline{x}, t)$ is the velocity field. This quantity is therefore carried by the turbulent fluid, but does not affect its motion. It is clear that if we take

$$\psi(\underline{x}, t_0) = \delta(\underline{x} - \underline{x}_0), \quad (2.3)$$

where $\delta(\underline{x})$ is the three dimensional Dirac δ -function, then

$$\psi(\underline{x}, t) = \delta(\underline{x} - \underline{x}_t), \quad (2.4)$$

where \underline{x}_t is the position at time t (in this particular realization of \underline{x}) of the fluid element which was initially at \underline{x}_0 . By averaging over all realizations, we see that the solution of

$$\frac{\partial}{\partial t} \langle \psi(\underline{x}, t) \rangle + \frac{\partial}{\partial \underline{x}_i} \langle \psi(\underline{x}, t) u_i(\underline{x}, t) \rangle = 0, \quad (2.5)$$

which satisfies the initial condition

$$\langle \psi(\underline{x}, t_0) \rangle = \delta(\underline{x} - \underline{x}_0), \quad (2.6)$$

is

$$\langle \psi(\underline{x}, t) \rangle = G(\underline{x}, t | \underline{x}_0, t_0). \quad (2.7)$$

Through a similar though more complicated equation, $\langle \psi u_i \rangle$ is related to higher moments, and so on. Again, we have an hierarchy of equations which must be closed in some way in order to evaluate $G(\underline{x}, t | \underline{x}_0, t_0)$.

In the present paper, we adopt the second of these two approaches and employ an approximation method for closing the equations devised by Kraichnan (1959). However, before presenting this approximate analysis, we shall derive some results which are asymptotically exact for small $t - t_0$.

2.2 Exact results for short times

For $t - t_0 \ll l/v_o$, the fluid particles are simply swept from their points of origin with whatever velocity the turbulent fluid happens to have at the moment of their release; i.e.

$$v(\underline{x}, \underline{u}, t | \underline{x}_0, t_0) = P[\underline{u}(\underline{x}_0, t_0)] \delta[(\underline{x} - \underline{x}_0) - \underline{u}(t - t_0)], \quad (2.8)$$

where $P[\underline{u}(\underline{x}_0, t_0)]$ is the probability distribution function (p.d.f.) of \underline{u} at position \underline{x}_0 and time t_0 . Thus, by (2.1), or by inspection

$$G(\underline{x}, t | \underline{x}_0, t_0) = \frac{1}{(t - t_0)^3} P\left(\frac{\underline{x} - \underline{x}_0}{t - t_0}\right). \quad (2.9)$$

This result can also be deduced from the formal solution

$$\begin{aligned}\psi(\underline{x}, t) &= \psi(\underline{x}, t_0) - \int_{t_0}^t dt' \frac{\partial}{\partial x_i} [u_i(\underline{x}, t') \psi(\underline{x}, t_0)] \\ &+ \int_{t_0}^t dt' \frac{\partial}{\partial x_i} \left\{ u_i(\underline{x}, t') \int_{t_0}^{t'} dt'' \frac{\partial}{\partial x_j} [u_j(\underline{x}, t'') \psi(\underline{x}, t_0)] \right\} - \dots, \quad (2.10)\end{aligned}$$

which one obtains from (2.2) by integration and iteration. When $t-t_0 \ll \ell/v_o$, then $\underline{u}(\underline{x}, t) \doteq \underline{u}(\underline{x}_o, t_o)$, and it follows that

$$\begin{aligned}\psi(\underline{x}, t) &\doteq \left[1 - (t-t_0) u_i(\underline{x}_o, t_o) \frac{\partial}{\partial x_i} + \frac{1}{2!} (t-t_0)^2 u_i(\underline{x}_o, t_o) u_j(\underline{x}_o, t_o) \frac{\partial^2}{\partial x_i \partial x_j} \right. \\ &\quad \left. - \dots \right] \psi(\underline{x}, t_o). \quad (2.11)\end{aligned}$$

Hence, by (2.6) and (2.7),

$$\begin{aligned}G(\underline{x}, t | \underline{x}_o, t_o) &\doteq \left[1 - (t-t_0) \left\langle u_i(\underline{x}_o, t_o) \right\rangle \frac{\partial}{\partial x_i} \right. \\ &\quad \left. + \frac{1}{2!} (t-t_0)^2 \left\langle u_i(\underline{x}_o, t_o) u_j(\underline{x}_o, t_o) \right\rangle \frac{\partial^2}{\partial x_i \partial x_j} - \dots \right] \delta(\underline{x} - \underline{x}_o). \quad (2.12)\end{aligned}$$

That this is equivalent to (2.9) can most easily be seen by expressing the result in wave-vector space, writing

$$G(\underline{x}, t | \underline{x}_o, t_o) = \int \widetilde{G}(\underline{k}, t | \underline{x}_o, t_o) e^{i \underline{k} \cdot (\underline{x} - \underline{x}_o)} d\underline{k}. \quad (2.13)$$

Then (2.12) is equivalent to

$$\tilde{G}(\underline{k}, t | \underline{x}_o, t_o) = \frac{1}{(2\pi)^3} \left[1 - ik_i(t-t_o) \langle u_i(\underline{x}_o, t_o) \rangle - \frac{1}{2!} k_i k_j (t-t_o)^2 \langle u_i(\underline{x}_o, t_o) u_j(\underline{x}_o, t_o) \rangle + \dots \right]; \quad (2.14)$$

$$\tilde{G}(\underline{k}, t | \underline{x}_o, t_o) = \frac{1}{(2\pi)^3} \tilde{P}[\underline{k}(t-t_o)], \quad (2.15)$$

where

$$\tilde{P}(\underline{\eta}) = \int P[u(\underline{x}_o, t_o)] e^{-i\underline{\eta} \cdot \underline{u}} d\underline{u} \quad (2.16)$$

is the characteristic function for the distribution of velocity at \underline{x}_o and t_o . Equation (2.9) is simply the inverse of (2.15).

This second method of establishing the behavior of G at small times brings out some noteworthy features. If (2.12) is cut off after any finite number of terms, it implies that $G(\underline{x}, t | \underline{x}_o, t_o)$ vanishes identically for non-zero $\underline{x}-\underline{x}_o$. On the other hand, if (2.14) is cut off after a finite number of terms, the resulting expression for \tilde{G} diverges for large \underline{k} , or large $t-t_o$. One concludes that any reasonable approximate solution for the full space-function $G(\underline{x}, t | \underline{x}_o, t_o)$ must include terms of orders of the formal expansion (2.10). Even for very short times the formal expansion is only useful because we happen to be able to sum it to all orders.

However, if the moments

$$\langle \Delta x_i \rangle = \int \Delta x_i G(x, t | x_o, t_o) dx = i(2\pi)^3 \left[\frac{\partial}{\partial k_i} \tilde{G}(k, t | x_o, t_o) \right]_{k=0}$$

$$\langle \Delta x_i \Delta x_j \rangle = \int \Delta x_i \Delta x_j G(x, t | x_o, t_o) dx = -(2\pi)^3 \left[\frac{\partial^2}{\partial k_i \partial k_j} \tilde{G}(k, t | x_o, t_o) \right]_{k=0}$$

($\Delta x = x - x_o$) are expanded by means of the formal solution, the resulting series appear to converge for all $t - t_o$, although the convergence is poor unless $t - t_o \ll \ell/v_o$. Equation (2.14) shows that for small $t - t_o$,

$$\langle \Delta x_i \Delta x_j \rangle = U_{ij}(x_o, t_o; x_o, t_o) (t - t_o)^2, \quad (2.17)$$

where

$$U_{ij}(x, t; x_o, t_o) = \langle u_i(x, t) u_j(x_o, t_o) \rangle$$

is the velocity covariance. In the isotropic case, therefore,

$$\langle \Delta x_i \Delta x_j \rangle = v_1^2 \delta_{ij} (t - t_o)^2, \quad (2.18)$$

where v_1^2 is the mean square of any component of velocity at position x_o and time t_o .

There is fairly strong experimental evidence (see, for example, Batchelor, 1953, Ch. 8) that P almost always is closely Gaussian. In this case, $G(x, t | x_o, t_o)$ must be closely Gaussian also for short times.

Then by (2.14) and (2.9) we have

$$\tilde{G}(k, t | x_o, t_o) = \frac{1}{(2\pi)^3} \exp \left[-\frac{1}{2} U_{ij}(x_o, t_o; x_o, t_o) k_i k_j (t - t_o)^2 \right], \quad (2.19)$$

and

$$G(x, t | x_o, t_o) = \frac{1}{(2\pi)^{3/2} (t - t_o)^3 (\det U_{ij})^{1/2}} \times \exp \left[-\frac{1}{2} u_{ij} \Delta x_i \Delta x_j (t - t_o)^{-2} \right], \quad (2.20)$$

where u_{ij} is the cofactor of $U_{ij}(x_o, t_o; x_o, t_o)$ and $\det U_{ij}$ denotes the determinant of these quantities. In the isotropic case,

$$\tilde{G}(k, t | x_o, t_o) = \frac{1}{(2\pi)^3} \exp \left[-\frac{1}{2} k^2 v_1^2 (t - t_o)^2 \right], \quad (2.21)$$

and

$$G(x, t | x_o, t_o) = \frac{1}{[2\pi v_1^2 (t - t_o)^2]^{3/2}} \exp \left[-(x - x_o)^2 / 2v_1^2 (t - t_o)^2 \right]. \quad (2.22)$$

2.3 An integro-differential equation for $G(x, t | x_o, t_o)$

When $t - t_o$ is not small compared to ℓ/v_o , the approximation (2.11) to the formal solution (2.10) is invalid since it is no longer legitimate to ignore the space and time variation of $u_i(x, t)$. It is nevertheless possible to effect a partial summation of (2.1) which includes terms from every order in the expansion, and is such that the resulting expression for \tilde{G} converges

for large k . The integral equation for this approximate form of G is derived in Appendix A in two ways. The first makes use of Kraichnan's direct interaction approximation (Kraichnan, 1959). In the second method, the same result is derived by discarding or retaining terms in the formal expansion according to a certain selection criterion. The terms retained are of all order. Each of these methods supposes that the velocity field is spatially homogeneous, but this restriction can be removed by the application of a more general method due to Kraichnan (1960). The final result, for the case where the mean field $\langle u(x, t) \rangle$ vanishes, is the equation

$$\begin{aligned} \frac{\partial}{\partial t} G(x, t | x_0, t_0) &= \int_{t_0}^t dt' \int dx' U_{ij}(x, t; x_0, t_0) \\ &\times \frac{\partial}{\partial x_i} G(x, t | x', t') \frac{\partial}{\partial x'_j} G(x', t | x_0, t_0). \end{aligned} \quad (2.23)$$

Here, as later, the fluid velocity is supposed incompressible:

$$\partial u_i(x, t) / \partial x_i = 0. \quad (2.24)$$

(The compressible case can be treated by similar methods.) In the case of statistically homogeneous and stationary flows, we may write

$$\left. \begin{aligned} G(x, t | x_0, t_0) &= G(x - x_0, t - t_0), \\ U_{ij}(x, t | x_0, t_0) &= U_{ij}(x - x_0, t - t_0), \end{aligned} \right\} \quad (2.25)$$

and, upon a partial integration, (2.23) becomes

$$\frac{\partial}{\partial t} G(x, t) = \frac{\partial^2}{\partial x_i \partial x_j} \int_0^t dt' \int dx' U_{ij}(x', t') G(x', t') G(x - x', t - t'). \quad (2.26)$$

Multiplying (2.26) by $x_i x_j$ and integrating the right-hand side by parts, we find

$$\frac{\partial}{\partial t} \langle x_i x_j \rangle = \int x_i x_j \frac{\partial}{\partial t} G(x, t) dx = 2 \int_0^t dt' \int dx' U_{ij}(x', t') G(x', t');$$

i.e.

$$\langle x_i x_j \rangle = 2t \kappa_{ij}(t), \quad (2.27)$$

where the variable "eddy diffusivity" tensor $\kappa_{ij}(t)$ is defined by

$$\kappa_{ij}(t) = \frac{1}{t} \int_0^t dt' (t - t') \int dx' U_{ij}(x', t') G(x', t'). \quad (2.28)$$

For short times ($t \ll \ell/v_o$), (2.27) and (2.28) agree with (2.17), if we assume $G(x', t')$ is negligible unless $|x'| \ll \ell$. Then,

$$\kappa_{ij} \doteq \frac{1}{2} t U_{ij}(0,0), \quad t \rightarrow 0. \quad (2.29)$$

For large times ($t \gg \ell/v_o$),

$$\kappa_{ij} \doteq \int_0^\infty dt' \int dx' U_{ij}(x', t') G(x', t') = \frac{1}{(2\pi)^3} \int_0^\infty dt' \int dk' \tilde{U}_{ij}(k, t') \tilde{G}(-k, t').$$

(2.30)

Let us assume that the diffusion for time $\gtrsim \ell/v_o$ is dominated by the energy-containing eddies. Then it is reasonable to suppose that the integral in (2.30) should depend only on the parameters ℓ and v_o , whence, by dimensional reasoning, we must have

$$\kappa_{ij} \rightarrow \text{constant of order } \ell v_o, \quad t \rightarrow \infty.$$

In the isotropic case, we have (setting $x = |x|$)

$$\kappa_{ij} = \kappa \delta_{ij}, \quad \kappa = \frac{4\pi}{3} \int_0^t dt' (t-t') \int_0^\infty dx' x'^2 U_{ii}(x', t') G(x', t').$$

(2.31)

Equations (2.27) and (2.28), and their generalizations for flow which are not steady or homogeneous, imply that the Lagrangian correlation function $\langle u_i(t)u_j(t_o) \rangle_L$ (cf. Taylor, 1921) is

$$\langle u_i(t)u_j(t_o) \rangle_L = \int dx U_{ij}(x, t; x_o, t_o) G(x, t | x_o, t_o).$$

(2.32)

This should be compared to the exact result

$$\langle u_i(t)u_j(t_o) \rangle_L = \int dx \langle u_i(x, t)u_j(x_o, t_o) \psi(x, t | x_o, t_o) \rangle,$$

(2.33)

where $\psi(x, t | x_0, t_0)$ denotes the unaveraged Green's function, i.e. the solution of (2.2) satisfying (2.3). Equation (2.32) clearly may be obtained from (2.33) on the assumption that the ψ and u fields are uncorrelated (see also Corrsin, 1960)*. However, if one decides to use an assumption of this kind, the results depend very much on what stage of the analysis is chosen for its application. For example, had we assumed that ψ and u are uncorrelated in (2.2), we would have obtained the absurd result $\partial \langle \psi(x, t) \rangle / \partial t = 0$. Moreover, if the approximation

$$\langle u_i(x, t) u_j(x_0, t_0) \psi(x, t | x_0, t_0) \rangle = U_{ij}(x, t; x_0, t_0) G(x, t | x_0, t_0) \quad (2.34)$$

is used to close the hierarchy of equations which arise from (2.2) and (2.3), the result is not consistent with (2.32) (cf. Roberts, 1957). Thus, the direct-interaction approximation is consistent with (2.34) only if (2.34) is used in a particular way.

2.4 Solution of the integro-differential equation for short times

For $t \ll \ell/v_0$, we may assume that the G factors in the integrand of (2.23) are negligible unless $x-x_0$ and $x'-x_0$ are both small compared with ℓ . Thus, $U_{ij}(x', t', x_0, t_0)$ in the integrand may be replaced by $U_{ij}(x_0, t_0, x_0, t_0)$ or, in the isotropic case, by $v_1^2 \delta_{ij}$. It follows that

$$\frac{\partial}{\partial t} G(x, t) = v_1^2 \nabla^2 \int_0^t dt' \int dx' G(x', t') G(x-x', t-t'). \quad (2.35)$$

* Saffman (unpublished) has used the approximate result (2.32) to derive an estimate for κ (see Sec. 2.5 below).

Taking the Fourier transform

$$G(x, t) = \int \tilde{G}(k, t) e^{ik \cdot x} dk,$$

we find

$$\frac{\partial}{\partial t} \tilde{G}(k, t) = -(2\pi)^3 v_1^2 k^2 \int_0^t dt' \tilde{G}(k, t') \tilde{G}(k, t-t'). \quad (2.36)$$

By taking the Laplace transform of (2.36), or by using a result of Watson (1944, Sec. 12.2, eq. 5), we find

$$\tilde{G}(k, t) = \frac{1}{(2\pi)^3} \frac{J_1(2kv_1 t)}{kv_1 t}, \quad (k = |k|). \quad (2.37)$$

Inverting by using a further result of Watson (1944, Sec. 13.42, eq. (4)), we find

$$G(x, t) = \begin{cases} \frac{1}{(2\pi v_1 t)^2} \left[4v_1^2 t^2 - x^2 \right]^{-1/2}, & \text{if } x < 2v_1 t, \\ 0 & \text{if } x > 2v_1 t. \end{cases} \quad (2.38)$$

The corresponding probability distribution $D(x_1, t)$ for displacement along the x_1 -axis (whose direction may be chosen arbitrarily) is related to $G(x, t)$ by

$$\partial D(x, t)/\partial x = -2\pi x G(x, t), \quad (2.39)$$

and so, in the present case,

$$D(x_1, t) = \begin{cases} \frac{1}{2\pi(v_1 t)^2} [4v_1^2 t^2 - x_1^2]^{1/2}, & \text{if } x_1 < 2v_1 t, \\ 0 & \text{if } x_1 > 2v_1 t. \end{cases} \quad (2.40)$$

Both $G(x, t)$ and $D(x_1, t)$ are everywhere non-negative, but they are zero beyond a distance of $2v_1 t$ from the source. That this behavior is not restricted to small times can be seen in a rough way from the fact that $k^{3/2} \tilde{G}(k, t)$ oscillates with wavelength $2v_1 t$ as $k \rightarrow \infty$. A more convincing demonstration can be given directly by induction from (2.26). One supposes that $G(x', t')$ vanishes when $|x'| > 2v_1 t'$ for all times $t' \leq t''$ and, by approximating to (2.26) for points near the wave-front $|x| = 2v_1 t''$, one can show that this implies that $G(x, t)$ vanishes for $|x| > 2v_1 t$, where t is infinitesimally greater than t'' . The finite maximum "propagation speed" exhibited by (2.38) therefore persists for all times. This is also consistent with (2.9) which implies, in the present case, that the p.d.f. of velocity has a sharp cut-off at $u = 2v_1$. These properties are physically unreasonable. They arise in some way from the direct interaction approximation.

A related unrealistic feature of (2.36) is that $G(x, t) \rightarrow \infty$ as $x \rightarrow 2v_1 t - 0$. However, the singularity is weak, since

$$\int_x^{2v_1 t + 0} 4\pi x^2 G(x, t) dx \doteq \frac{4}{\pi} \left(2 - \frac{x}{v_1 t}\right)^{1/2}, \quad x \rightarrow 2v_1 t - 0;$$

i.e. the singular outgoing wave front does not even carry a finite integrated probability. It is perhaps worth mentioning that this singular type of behavior is not unknown in random walk problems although it is generally smoothed away after a few steps. (For random walk in a plane, the distribution after two steps shows singularities, see Watson Sec. 13.48; Sec. 13.46, eq. (3); for random walk on a sphere similar results hold, see Roberts and Ursell, 1959, p. 320, 2nd footnote.)

For a normally distributed velocity field, a comparison of (2.37) in the form

$$\tilde{G}(\underline{k}, t) = \frac{1}{(2\pi)^3} \sum_{n=0}^{\infty} \frac{(-1)^n (kv_1 t)^{2n}}{n! (n+1)!}, \quad (2.41)$$

with the exact solution (2.21) in the form

$$\tilde{G}(\underline{k}, t) = \frac{1}{(2\pi)^3} \sum_{n=0}^{\infty} \frac{(-1)^n (kv_1 t)^{2n}}{2^n n!}, \quad (2.42)$$

shows that, for short times, the direct interaction approximation gives the second moments of $G(\underline{x}, t)$ correctly. The fractional errors in the fourth, sixth, and $2n^{\text{th}}$ moments are, respectively, $1/3$, $2/3$, and $[1 - 2^n/(n + 1)!]$. This exhibits in another way the consequences of the effective cut-off in "propagation speed".

2.5 Solution for large times

Fot $t \gg \ell/v_o$, the diffusing particles will have suffered many displacements (statistically almost unrelated) from the energy containing eddies, and we may expect $G(\underline{x}, t)$ to become close to a Gaussian distribution. This may be

established directly (cf. Taylor, 1921) by expressing the moments of the distribution as integrals over Lagrangian velocity correlations. It can then be shown that, for times large compared to the Lagrangian correlation times (assuming these to be finite), these moments approach Gaussian values corresponding to an eddy diffusivity

$$\kappa_{ij} = \int_0^\infty dt' \langle u_i(t') u_j(t_o) \rangle_L . \quad (2.43)$$

In a like manner, for $t \gg l/v_o$, the distribution satisfying (2.26) becomes Gaussian with an eddy diffusivity

$$\kappa_{ij} = \int_0^\infty dt' \int dx' G(x', t') U_{ij}(x', t') . \quad (2.44)$$

One way of proving this is to extend to the higher moments the arguments that led to (2.27). For example, it is easy to show from (2.26) that

$$\begin{aligned} \langle x_i x_j x_k x_l \rangle &= \langle x_i x_j \rangle \langle x_k x_l \rangle + \langle x_i x_k \rangle \langle x_j x_l \rangle + \langle x_i x_l \rangle \langle x_j x_k \rangle \\ &+ 2 \left[\int_0^t dt' (t-t') \int dx' x_i' x_j' U_{kl}(x', t') G(x', t') + 5 \text{ similar terms} \right]. \end{aligned} \quad (2.45)$$

Assuming that expressions such as

$$\frac{1}{t} \int_0^t dt' (t-t') \int dx' x_i' x_j' U_{kl}(x', t') G(x', t')$$

converge as $t \rightarrow \infty$, we see that for $t \gg \ell/v_o$ the first three terms on the right-hand side of (2.45) are of order $(\ell v_o t)^2$ while the remainder are of order $\ell^3 v_o^3 t$ and therefore are comparatively negligible. Thus the fourth order cumulants can be neglected if $t \gg \ell/v_o$. Similar arguments hold in all orders and show that G approaches normality. This result may also be established by the following alternative method.

If $t \gg \ell/v_o$, the mean distance $\sqrt{\langle x^2 \rangle}$ the particles will have travelled from their source and will be large compared to ℓ so that the length and time scales of $G(x, t)$ will be large compared to ℓ and ℓ/v_o , respectively. Under these circumstances, the only regions of integration in (2.26) for which the integrand is appreciable are those for which $G(x-x', t-t')$ is approximately equal to $G(x, t)$. Thus, with κ_{ij} given by (2.44), we have

$$\frac{\partial G(x, t)}{\partial t} = \kappa_{ij} \frac{\partial^2 G(x, t)}{\partial x_i \partial x_j}, \quad (2.46)$$

since for such large times it is immaterial whether the upper limit of integration in (2.26) is t or ∞ . In the isotropic case (2.46) becomes

$$\frac{\partial G(x, t)}{\partial t} = \kappa \nabla^2 G(x, t), \quad (2.47)$$

where κ is given by (2.31) with t replaced by ∞ .

We may regard (2.46) as the first term in a series of approximations based on expanding the term $G(x-x', t-t')$ in the integrand of (2.26) in a Taylor series about the point x, t , and we may apply an a posteriori check on the reasonableness of the approximation by verifying that, on substituting in the second term of this series the value of $G(x, t)$ derived

from (2.46), this second term is small compared to either side of (2.46). We will refrain from giving the analysis, which is straightforward. It confirms that, if the relevant integrals converge, the necessary conditions for the validity of (2.46) are

$$t \gg \ell/v_0 \quad , \quad x \ll v_0 t. \quad (2.48)$$

The second condition arises from the finiteness of the maximum propagation velocity which, as we have already seen, gives an artificial cut-off at the distance $2v_1 t$. We may expect that for the actual case in which this cut-off is not present the second of the conditions (2.48) would be unnecessary and that the distribution of particles would be Gaussian at all distances. In any event, when the first condition is satisfied, it is clear that the fraction of particles affected by the second condition is negligibly small.

It seems extremely likely, from the exact short-time result of Sec. 2.2 and our present results for large times, that $G(x, t)$ is nearly Gaussian for all times, and that the variable diffusion coefficients defined in Sec. 2.3 (cf. eqs. 2.27, 2.28, 2.31) will give useful estimates of the variance of the distribution at all times. On assuming a Gaussian form for $G(x, t)$, we obtain from (2.28) an integral equation for $\kappa_{ij}(t)$. The approximation of (2.33) by (2.32) has been proposed independently by P.G. Saffman (unpublished) in the homogeneous case. By assuming a Gaussian form for $G(x, t)$, and the isotropic form

$$\tilde{U}_{ij}(k, t) = \left[\delta_{ij} - \frac{k_i k_j}{k^2} \right] \frac{v_1^2 k^2}{\pi^{3/2} k_o^5} \exp \left\{ \left[\frac{k^2}{k_o^2} + \frac{1}{2} v_1^2 k^2 t^2 \right] \right\}, \quad (2.49)$$

he has obtained from the integral equation for κ_{ij} the estimate

$$\kappa \sim 0.7 v_1 / k_o, \quad (t \rightarrow \infty), \quad (2.50)$$

where $\pi^{1/2} k_o^{-1}$ is the longitudinal integral scale (cf. Batchelor, 1953, p.47).

3. RELATIVE DIFFUSION

3.1 Formulation of problem: exact results for short times

In this section, we study the correlation between the motion of two marked particles which are initially separated by a distance small compared to ℓ . The choice of method is essentially that of Sec. 2.1 and again we will adopt a formulation of terms of Eulerian moments. We introduce the passive scalar field $\psi_1(x, t)$ for the first particle and $\psi_2(y, s)$ for the second particle, and we require that both fields satisfy (2.2). For $\psi_1(x, t)$, we take as initial condition

$$\psi_1(x, t_o) = \delta(x - x_o), \quad (3.1)$$

and, for $\psi_2(x, t)$, we take

$$\psi_2(y, s_o) = \delta(y - y_o). \quad (3.2)$$

Let us define a two particle Green's function by

$$G(x, t; y, s | x_0, t_0; y_0, s_0) = \langle \psi_1(x, t) \psi_2(y, s) \rangle. \quad (3.3)$$

Clearly, since the two particles have identical properties,

$$G(x, t; y, s | x_0, t_0; y_0, s_0) = G(y, s; x, t | y_0, s_0; x_0, t_0).$$

In homogeneous steady turbulence, it depends only on difference times and difference coordinates. We shall then write it as

$$G(x-x_0, t-t_0; y-y_0, s-s_0 | r_0, \tau_0) = G(y-y_0, s-s_0; x-x_0, t-t_0 | -r_0, -\tau_0)$$

where $r_0 = y_0 - x_0$ and $\tau_0 = s_0 - t_0$.

It is evident that if one integrates (3.3) over all x or all y the one-point Green's function of Sec. 2 is recovered. Of more interest is the integral

$$R(r, t, s | x_0, r_0, t_0, s_0) = \int dx G(x, t; x+r, s | x_0, t_0; x_0+r_0, s_0). \quad (3.4)$$

$R(r, t, s | x_0, r_0, t_0, s_0)$ is Richardson's "distance neighbour function" (Richardson, 1926). It denotes the p.d.f. at time t of separation r for a pair of particles which at time t_0 were situated at x_0 and $x_0 + r_0$. For the homogeneous and stationary flows with which we will be primarily concerned, R depends only on $r_0, \tau_0, r-r_0, t-t_0$, and $s-s_0$ and will be written

$$R(r-r_0, t-t_0, s-s_0 | r_0, \tau_0) = \int dx G(x-x_0, t-t_0; (x-x_0)+(r-r_0), s-s_0 | r_0, \tau_0). \quad (3.5)$$

Let us now assume that the Reynolds number of the flow is sufficiently high that an inertial range of wave numbers, or eddy sizes, exists. By this we mean that the wave numbers which contain most of the energy are distinct from the higher wave numbers which are responsible for most of the energy dissipation. Suppose now, that r_o lies with this inertial range of eddy sizes. The eddies of dimension large compared to r_o move the two marked particles together bodily without substantially altering the magnitude or direction of r_o . In a frame of reference moving with these large scale motions, the eddies of dimension small compared to r_o are associated with a small r.m.s. velocity and have little effect upon r_o . The rate of separation of the particles, in this case, is dominated by eddies of dimension $\sim r_o$, because such eddies make the principal contribution to the relative velocity of two points separated by a distance r_o (cf. Batchelor, 1953, Ch. 6). These eddies disperse the particles substantially in a time of order

$$T(r_o) = r_o \left[v_o^2 - u_{ii}(r_o, 0) \right]^{-1/2} \ll \ell/v_o. \quad (3.6)$$

For times short compared to $T(r_o)$, we may apply arguments similar to those of Sec. 2. These show that (cf. eq. 2.9)

$$G(x, t; y, s | x_o, t_o; y_o, s_o) = \frac{1}{(t-t_o)^{\frac{1}{3}}(s-s_o)^{\frac{1}{3}}} P\left(\frac{x-x_o}{t-t_o}, \frac{y-y_o}{s-s_o}\right), \quad (3.7)$$

where $P[u_1(x_o, t_o), u_2(y_o, s_o)]$ is the joint p.d.f. for velocity u_1 at position x_o and time t_o and velocity u_2 at position y_o and time s_o . It follows

(cf. eq. 2.17) that

$$\begin{aligned} \langle \Delta x_i \Delta y_j \rangle &= \iint \Delta x_i \Delta y_j G(x, t; y, s | x_o, t_o; y_o, s_o) dx dy, \\ &= 2U_{ij}(x_o, t_o; y_o, s_o)(t - t_o)(s - s_o), \end{aligned}$$

where $\Delta x = x - x_o$ and $\Delta y = y - y_o$. Also, for these short times, (3.7) shows that Richardson's function is

$$R(r, t, t | x_o, r_o, t_o, t_o) = \frac{1}{(t - t_o)^3} \mathcal{P}\left(\frac{r - r_o}{t - t_o}\right), \quad (3.9)$$

where $\mathcal{P}[v(r_o, t_o)]$ is the p.d.f. of relative velocity v between the two points x_o and $x_o + r_o$ at time t_o , (cf. Batchelor, 1952a, Sec. 3). It follows from (3.9) that

$$\langle \Delta r_i \Delta r_j \rangle = \langle v_i v_j \rangle (t - t_o)^2, \quad (3.10)$$

which, for an isotropic field with r_3 -axis along r_o , gives

$$\left. \begin{aligned} \langle (\Delta r_1)^2 \rangle &= \langle (\Delta r_2)^2 \rangle = 2v_1^2 [1 - g(r_o, t_o)] (t - t_o)^2, \\ \langle (\Delta r_3)^2 \rangle &= 2v_1^2 [1 - f(r_o, t_o)] (t - t_o)^2, \end{aligned} \right\} \quad (3.11)$$

where $f(r, t)$ and $g(r, t) = f(r, t) + \frac{1}{2} r \partial f(r, t) / \partial r$ are respectively the longitudinal and transverse velocity correlations at time t for points separated by a distance r . Thus, Richardson's function is initially oblate spheroidal with the line joining the origin ($r = 0$) to r_o as axis.

In the particular case in which $P(u_1, u_2)$ is stationary, isotropic and Gaussianly distributed at x_o and y_o , the double-Fourier transform

$$\tilde{G}(k, t; \ell, s | r_o, \tau_o) = \frac{1}{(2\pi)^6} \iint e^{-i(k \cdot x + \ell \cdot y)} G(x, t; y, s | r_o, \tau_o) dx dy \quad (3.12)$$

takes the form

$$\tilde{G}(k, t; \ell, s | r_o, \tau_o) = \frac{1}{(2\pi)^6} \exp \left\{ -\frac{1}{2} \left[v_1^2 k^2 t^2 + v_1^2 \ell^2 s^2 + 2U_{ij}(r_o, \tau_o) k_i \ell_j ts \right] \right\}. \quad (3.13)$$

It follows that Richardson's function

$$R(r, t, t | r_o, 0) = (2\pi)^3 \int \tilde{G}(-k, t; k, t | r_o, 0) e^{ik \cdot r} dk \quad (3.14)$$

then takes the form

$$R(r, t, t | r_o, 0) = \frac{1}{\pi^{3/2} (2v_1 t)^3 [1-g(r_o)] [1-f(r_o)]^{1/2}} \times \exp \left\{ -\frac{1}{(2v_1 t)^2} \left[\frac{(r_1^2 + r_2^2)}{[1-g(r_o)]} + \frac{r_3^2}{[1-f(r_o)]} \right] \right\}, \quad (3.15)$$

where, again, the r_3 -axis is along r_o .

3.2 Integral equations for $G(x, t; y, s | x_o, t_o; y_o, s_o)$

When $t - t_o$ and $s - s_o$ are not small compared to $T(r_o)$, it is no longer legitimate to ignore the spatial and temporal variations of $u_i(x, t)$. It is nevertheless possible to effect a partial summation of the formal solution for G containing terms of all orders in the expansion. Since the determination of G is a problem which is essentially inhomogeneous (even if the velocity field is homogeneous), the Fourier modes are not weakly dependent (see Appendix A), and the methods expounded by Kraichnan (1959) are not applicable. However, Kraichnan (1960) has recently generalized his methods to inhomogeneous problems, and has given an approximate equation of motion for the covariance $\langle \psi(x, t)\psi(x', t') \rangle$ of a convected passive scalar field. A straightforward generalization to our case of two scalar fields yields the result

$$\begin{aligned}
& \frac{\partial}{\partial t} G(x, t; y, s | x_o, t_o; y_o, s_o) \\
&= \int_{t_o}^t dt' \int dx' U_{ij}(x, t; x', t') \frac{\partial G(x, t | x', t')}{\partial x_i} \frac{\partial G(x', t'; y, s | x_o, t_o; y_o, s_o)}{\partial x'_j} \\
&+ \int_{s_o}^s ds' \int dy' U_{ij}(x, t; y', s') G(y, s | y', s') \frac{\partial^2 G(x, t; y', s' | x_o, t_o; y_o, s_o)}{\partial x_i \partial y'_j},
\end{aligned} \tag{3.16}$$

where $G(x, t | x_o, t_o)$ is the one-point Green's function of Sec. 2. An equation similar to (3.16) holds for $\partial G / \partial s$. The equation for $R(r, t, s | x_o, r_o, t_o, s_o)$ that can be obtained from (3.16) by integration may also, in the case of homogeneous velocity fields, be derived quite easily by a device which reduces

the problem to a fully homogeneous one in which the Fourier modes are weakly dependent. This matter is dealt with briefly in Appendix B. For velocity fields which are homogeneous and stationary, (3.16) may be written, after partial integration, in the form

$$\begin{aligned} \frac{\partial G(x, t; y, s | r_o, \tau_o)}{\partial t} &= \frac{\partial^2}{\partial x_i \partial x_j} \int_0^t dt' \int dx' U_{ij}(x', t') G(x', t') G(x - x', t - t'; y, s | r_o, \tau_o) \\ &+ \frac{\partial^2}{\partial x_i \partial y_j} \int_0^s ds' \int dy' U_{ij}(r_o - x + y - y', \tau_o - t + s - s') G(y', s') G(x, t; y - y', s - s' | r_o, \tau_o). \end{aligned} \quad (3.17)$$

The equation for $R(r, t, s | r_o, \tau_o)$ to which we referred above is

$$\begin{aligned} \frac{\partial R(r, t, s | r_o, \tau_o)}{\partial t} &= \frac{\partial^2}{\partial r_i \partial r_j} \left[\int_0^t dt' \int dx' U_{ij}(x', t') G(x', t') R(r - x', t - t', s | r_o, \tau_o) \right. \\ &\quad \left. - \int_0^s ds' \int dy' U_{ij}(r_o + r - y', \tau_o + \tau - s') G(y', s') R(r - y', t, s - s' | r_o, \tau_o) \right]. \end{aligned} \quad (3.18)$$

On multiplying this equation by $r_i r_j$ and integrating by parts, we find

$$\begin{aligned} \frac{\partial}{\partial t} \int R(r, t, s | r_o, \tau_o) r_i r_j dr &= 2 \int_0^t dt' \int dx' U_{ij}(x', t') G(x', t') \\ &\quad - 2 \int_0^s ds' \int dr U_{ij}(r_o + r, \tau_o + s' - t) R(r, t, s' | r_o, \tau_o). \end{aligned} \quad (3.19)$$

By differentiating (3.19) with respect to s , we find

$$\frac{\partial^2}{\partial t \partial s} \int R(r, t, s | r_0, \tau_0) r_i r_j dr = -2 \int dr U_{ij}(r_0 + r, \tau_0 + \tau) R(r, t, s | r_0, \tau_0). \quad (3.20)$$

Let us take $\tau_0 = 0$, $s_0 = 0$. Then, by (3.20) and (2.32), we see that the Lagrangian correlation for relative velocity V between two particles which were initially separated by a distance r_0 is

$$\langle v_i(t) v_j(s) \rangle_L = 2 \int dr U_{ij}(r, s-t) \left[G(r, s-t) - R(r-r_0, t, s | r_0, 0) \right]. \quad (3.21)$$

This result should be compared to the exact result

$$\begin{aligned} \langle v_i(t) v_j(s) \rangle_L &= \iint dx dy \left[\langle \psi_1(x, t) \psi_1(y, s) u_i(x, t) u_j(y, s) \rangle \right. \\ &\quad + \langle \psi_2(x, t) \psi_2(y, s) u_i(x, t) u_j(y, s) \rangle \\ &\quad - \langle \psi_1(x, t) \psi_2(y, s) u_i(x, t) u_j(y, s) \rangle \\ &\quad \left. - \langle \psi_2(x, t) \psi_1(y, s) u_i(x, t) u_j(y, s) \rangle \right], \end{aligned}$$

where $\psi_1(x, t) = \psi_1(x, t | x_0, 0)$ and $\psi_2(y, s) = \psi_2(y, s | y_0, 0)$ denote the unaveraged Green's functions. In stationary homogeneous flows, this result reduces to (3.21) on the assumption that the ψ and u fields are uncorrelated (cf. Sec. 2.3).

Also, by adding to (3.19) to the analogous equation for the derivative with respect to s , and setting $t = s$, we find

$$\begin{aligned} \frac{\partial}{\partial t} \langle r_i r_j \rangle &= \int \frac{\partial}{\partial t} R(r, t, t | r_o, 0) r_i r_j dr = \\ &= 4 \int_0^t dt' \int dx' U_{ij}(x', t') G(x', t') \\ &- 2 \int_0^t dt' \int dr U_{ij}(r_o + r, t - t') [R(r, t, t' | r_o, 0) + R(r, t', t | r_o, 0)]. \quad (3.22) \end{aligned}$$

For $t \ll T(r_o)$, R is negligible except near $r = 0$, and the right-hand side of (3.22) becomes

$$4 [U_{ij}(0, 0) - U_{ij}(r_o, 0)] t,$$

in agreement with (3.10). For $t \gg \ell/v_o$, R is appreciable even at separations r of order ℓ . For such large separations, the second term on the right-hand side of (3.22) is quite negligible, and we find (cf. eqs. 2.27, 2.28)

$$\langle r_i r_j \rangle = 2 \langle x_i x_j \rangle = 2 \langle y_i y_j \rangle = 4 \kappa_{ij}. \quad (3.23)$$

This is consistent with the intuitive notion that, at such large separations from their source and each other, the particles will wander independently as

in a true Brownian motion. For intermediate ranges of t , it appears that no such definite statements can be made. However a reasonable approximation appears to be possible and this is discussed in Sec. 3.4.

3.3 Solution of equations for short times

For $t \ll T(r_o)$, we may assume that the G factors in the integrands of (3.17) are both negligible unless x', y', x, y are small compared with r_o . Then, the U_{ij} factors in the integrands may be replaced by their values for zero x', y', x, y , and similarly for the time arguments. Therefore, in isotropic flows, we have

$$\begin{aligned} \frac{\partial G(x, t; y, s | r_o, \tau_o)}{\partial t} &= v_l^2 \delta_{ij} \frac{\partial^2}{\partial x_i \partial x_j} \int_0^t dt' \int dx' G(x', t') G(x-x', t-t'; y, s | r_o, \tau_o) \\ &- U_{ij}(r_o, \tau_o) \frac{\partial^2}{\partial x_i \partial y_j} \int_0^s ds' \int dy' G(y', s') G(x, t; y-y', s-s' | r_o, \tau_o). \end{aligned} \quad (3.24)$$

Take a combined Laplace and Fourier transform defined by

$$\tilde{G}^*(k, p; \ell, q | r_o, \tau_o) = \frac{1}{(2\pi)^6} \int_0^\infty dt e^{-pt} \int_0^\infty ds e^{-qs} \int dx e^{ik \cdot x} \int dy e^{i\ell \cdot y} G(x, t; y, s | r_o, \tau_o). \quad (3.25)$$

Then, by (3.24),

$$\begin{aligned}
& p \tilde{G}^*(k, p; \ell, q | r_o, \tau_o) - \frac{1}{(2\pi)^3} \tilde{G}^*(\ell, q) = - (2\pi)^3 k^2 v_1^2 \tilde{G}^*(k, p) \tilde{G}^*(k, p; \ell, q | r_o, \tau_o) \\
& - (2\pi)^3 U_{ij}(r_o, \tau_o) k_i \ell_j \tilde{G}^*(\ell, q) \tilde{G}^*(k, p; \ell, q | r_o, \tau_o), \tag{3.26}
\end{aligned}$$

where $\tilde{G}^*(k, p)$ is the Fourier transform with respect to x and the Laplace transform with respect to t of $G(x, t)$. By (2.34), we have

$$p + (2\pi)^3 k^2 v_1^2 \tilde{G}^*(k, p) = \frac{1}{(2\pi)^3 \tilde{G}^*(k, p)}. \tag{3.27}$$

By (3.26) and (3.27), it follows that

$$\tilde{G}^*(k, p; \ell, q | r_o, \tau_o) = \tilde{G}^*(k, p) \tilde{G}^*(\ell, q) / \left[1 + (2\pi)^6 U_{ij}(r_o, \tau_o) k_i \ell_j \tilde{G}^*(k, p) \tilde{G}^*(\ell, q) \right]. \tag{3.28}$$

If we expand the denominator of (3.28), we obtain

$$\tilde{G}^*(k, p; \ell, q | r_o, \tau_o) = \sum_{m=0}^{\infty} (-1)^m \left[(2\pi)^6 U_{ij}(r_o, \tau_o) k_i \ell_j \right]^m \left[\tilde{G}^*(k, p) \tilde{G}^*(\ell, q) \right]^{m+1}. \tag{3.29}$$

Now, by (3.27)

$$\tilde{G}^*(k, p) = 1 / \left\{ 4\pi^3 \left[p + \sqrt{p^2 + 4k^2 v_1^2} \right] \right\}. \tag{3.30}$$

The inverse Laplace transformation of $[\tilde{G}^*(k, p)]^{m+1}$ is therefore (see, for example, Watson, 1944, Sec. 13.2, eq. 7)

$$\frac{(m+1) J_{m+1} (2kv_1 t)}{(2\pi)^{3m+3} (kv_1)^{m+1} t} .$$

Thus

$$\begin{aligned} \tilde{G}(k, t; \ell, s | r_o, \tau_o) &= \frac{1}{(2\pi)^6} \sum_{o=0}^{\infty} (-1)^m (m+1)^2 \left[\frac{k_i \ell_j U_{ij}(r_o, \tau_o)}{kv_1^2} \right]^m \\ &\times \frac{J_{m+1}(2kv_1 t)}{kv_1 t} \frac{J_{m+1}(2\ell v_i s)}{\ell v_i s} . \end{aligned} \quad (3.31)$$

The second moments of (3.31) agree with (3.8); for a normally distributed velocity distribution, the higher moments are given with progressively less accuracy (cf. Sec. 2.4). In Appendix B, the significance of (3.31) in terms of diagrams is discussed. The relationship

$$\int \frac{J_{m+1}(2kv_1 t)}{(kv_1 t)^{m+1}} e^{ik \cdot x} dx = \begin{cases} \frac{2\pi m!}{(2m)!} \frac{(4v_1^2 t^2 - x^2)^{m-1/2}}{(v_1 t)^{2m+2}} & , \text{ if } x < 2v_1 t \\ 0 & , \text{ if } x > 2v_1 t \end{cases}$$

can be obtained by induction from results of Watson (1944, Sec. 13.42, eq. 4, $\mu = 1$, using also Sec. 13.24, eq. 1) or directly (Watson, Sec. 13.14). This enables us to invert the function $\tilde{G}(k, t; \ell, s | r_o, \tau_o)$ in the form (cf. eq. 2.36)

$$G(x, t; y, s | r_o, \tau_o) = \begin{cases} \frac{1}{(2\pi)^{\frac{1}{4}} (v_1 t)^{\frac{3}{2}} (v_1 s)^{\frac{3}{2}}} \sum_{m=0}^{\infty} (-1)^m \frac{[(m+1)!]^2}{[(2m)!]^2} \left[U_{ij}(r_o, \tau_o) \frac{\partial^2}{\partial x_i \partial y_j} \right]^m \\ \times \left[\left(1 - \frac{x^2}{4v_1^2 t^2} \right) \left(1 - \frac{y^2}{4v_1^2 s^2} \right) \right]^{m-1/2} \\ , \quad \text{if } x < 2v_1 t \text{ and } y < 2v_1 s, \\ 0 \\ , \quad \text{if } x > 2v_1 t \text{ or } y > 2v_1 s. \end{cases} \quad (3.32)$$

As in Sec. 2.3, the effect of the artificial cut-off $2v_1$ in the p.d.f. of velocity is apparent. It is also clear that, if $U_{ij}(r_o, \tau_o)$ is small compared to v_1^2 , (3.32) reduces to its first term

$$G(x, t; y, s | r_o, \tau_o) \doteq G(x, t)G(y, s),$$

where the right-hand side is given by (2.36). If the initial separation of the points is so close in space and time that we can write $U_{ij}(r_o, \tau_o) = v_i^2 \delta_{ij}$, equations (3.14) and (3.31) show (cf. Watson Sec. 11.41, eq. (12)) that

$$\tilde{R}(k, t, s | r_o, \tau_o) = (2\pi)^{\frac{3}{2}} \tilde{G}(-k, t; k, s | r_o, \tau_o) = \frac{1}{(2\pi)^{\frac{3}{2}}} \frac{J_1[2kv_1(s-t)]}{kv_1(s-t)},$$

or, inverting,

$$R(r, t, s | r_0, \tau_0) \rightarrow G(r_0 + r, s - t); \quad r_0 \rightarrow 0, \tau_0 \rightarrow 0. \quad (3.33)$$

This shows, as we expect, that in the limit $r_0 \rightarrow 0, \tau_0 \rightarrow 0$, the particles are not separated by the flow and simply move together as one particle. (As a further consistency check, we note that the Lagrangian correlation (3.21) is given correctly by (3.33) in this case.)

3.4 Solutions for large and intermediate times

For very long times, $t \gg \ell/v_0$, there is a high probability that the particles have separated by a distance comparable to, or greater than, ℓ and therefore wander independently. In fact, the solution of (3.17) is exactly analogous to that of Sec. 2.5 for the one-point Green's function, and we find

$$G(x, t; y, s | r_0, \tau_0) \rightarrow G(x, t)G(y, s), \quad (r \gg \ell), \quad (3.34)$$

where $G(x, t)$ and $G(y, s)$ are given, in homogeneous flows, by the appropriate solutions of (2.45). In this case we find $R(r, t)$ satisfies

$$\frac{\partial R(r, t)}{\partial t} = 2 \kappa_{ij} \frac{\partial^2 R(r, t)}{\partial r_i \partial r_j},$$

so that $R(r, t)$ assumes a Gaussian form corresponding to a diffusivity twice that characteristic of the one particle Green's function (cf. eqs. 2.45, 3.23).

Consider turbulence at high Reynolds numbers. The existence of an inertial range implies that the spectrum $E(k)$ obeys

$$\int_0^k E(k) dk \doteq \frac{1}{2} v_o^2, \quad \text{if } k \gg k_o,$$

$$\int_k^\infty E(k) k^2 dk \doteq \frac{1}{2} \epsilon / \nu, \quad \text{if } k \ll k_d,$$

where $k_o = (1/\ell)$ and k_d are wave numbers characteristic of the energy containing range and the dissipation range, respectively. (ϵ = rate of dissipation per unit mass; ν = kinematical viscosity.) It follows that

$$kE(k) \rightarrow 0, \quad k \rightarrow \infty; \quad k^3 E(k) \rightarrow 0, \quad k \rightarrow 0.$$

For the later developments (cf. eqs. 3.53, 3.54 below), we will require the more stringent conditions

$$kE(k) \rightarrow 0, \quad k \rightarrow \infty; \quad k^2 E(k) \rightarrow 0, \quad k \rightarrow 0. \quad (3.35)$$

If the initial separation satisfies

$$\ell \gg r_o \gg 1/k_d, \quad (3.36)$$

we expect that, in a time large compared to $T(r_o)$ but small compared to ℓ/v_o , $\langle (r-r_o)^2 \rangle$ will become large compared to r_o^2 but remain small compared to ℓ^2 . For these "intermediate times" (as we shall term them), neither the short-time solution of Sec. 3.3 nor the long-time solution above is valid.

Consider $R(r-r_0, t, t+\tau | r_0, 0)$ for $\tau > 0$. This quantity is the p.d.f. of the separation of two particles (released at $t = 0$ at a separation of r_0) one of which has been carried by the flow for a time t , and the other for a time $t+\tau$. This process may be visualised in two stages. During the first, of duration t , both particles are carried by the flow. Their separation r during this time is essentially unaffected by the energy-containing eddies which give nearly equal displacements to both particles. It is governed by the motions (relative to the energy containing eddies) of dimension $\sim r$. Now, in a frame moving with the energy-containing eddies, the r.m.s. velocity associated with these small scale motions is very small compared to v_0 . Thus, the mean square separation $\langle r^2 \rangle$ at the termination of the first stage is very small compared to $(v_0 t)^2$. During the second stage, of duration τ , one of the particles can be considered as fixed in space while the other is carried by the flow for a further time τ . Its motion during this time is dominated by the energy-containing motions. During the first stage the relative diffusion is given by $R(r, t, t | r_0, 0)$. During the second stage the further diffusion of the second particle should be given by $G(r, \tau)$, since the energy-containing motions are almost uncorrelated with the small scale motions. Thus, we expect

$$R(r, t, t+\tau | r_0, 0) \doteq \int R(r', t, t | r_0, 0) G(r-r', \tau) dr'. \quad (3.37)$$

It is clear that this result is exact for $t = 0$ or $\tau = 0$. Also, the change in the mean square separation during the second stage will be of the order of $(v_0 \tau)^2$. Thus, when this is large compared to the value of $\langle r^2 \rangle$ at the end of the first stage, (3.37) becomes

$$R(r, t, t+\tau | r_o, 0) \doteq G(r, \tau). \quad (3.38)$$

Consider now (3.18) and its counterpart for $\partial R / \partial s$. These show that

$$\begin{aligned} \frac{\partial R(r, t, t | r_o, 0)}{\partial t} &= \frac{\partial^2}{\partial r_i \partial r_j} \left\{ \int_0^t dt' \int dr' \left[U_{ij}(r', t') - U_{ij}(r_o + r - r', t') \right] \right. \\ &\quad \times G(r', t') \left. \left[R(r - r', t - t', t | r_o, 0) + R(r - r', t, t - t' | r_o, 0) \right] \right\}. \end{aligned} \quad (3.39)$$

By the arguments above, the integrand should be dominated by contributions from small t' (because of the behavior of R) and from small r' (because of the behavior of G). We will therefore replace the R factors in the integrand by $R(r, t, t | r_o, 0)$. It seems likely from the preceding discussion that this approximation will lead to results which are at least qualitatively correct.

We now find

$$\frac{\partial R(r, t, t | r_o, 0)}{\partial t} = \frac{\partial^2}{\partial r_i \partial r_j} \left[K_{ij}(r, t) R(r, t, t | r_o, 0) \right], \quad (3.40)$$

where

$$K_{ij}(r, t) = 2 \int_0^t dt' \int dr' \left[U_{ij}(r', t') - U_{ij}(r_o + r - r', t') \right] G(r', t'). \quad (3.41)$$

For short times $t \ll T(r_o)$, (3.41) becomes

$$K_{ij}(r, t) = 2t \left[v_1^2 \delta_{ij} - U_{ij}(r_o, 0) \right], \quad (3.42)$$

which, by (3.40), is in agreement with (3.10). For very large times $t \gg \ell/v_o$, there is a high probability that the separation of particles is $\gtrsim \ell$. For these separations, the second U factor in the integrand of (3.41) is negligible, and the first factor gives

$$K_{ij}(r, t) = 2 \kappa_{ij}, \quad (3.43)$$

in agreement with the results derived earlier (cf. 3.23).

To calculate the form of $K_{ij}(r, t)$ for intermediate times, we express (3.41) in the form

$$K_{ij}(r, t) = 2(2\pi)^3 \int_0^t dt' \int dk \left[1 - e^{ik \cdot (r+r_o)} \right] \tilde{U}_{ij}(k, t') \tilde{G}(k, t'). \quad (3.44)$$

Assuming isotropy, we can write

$$\tilde{U}_{ij}(k, t) = \left(\delta_{ij} - \frac{k_i k_j}{k^2} \right) \frac{E(k, t)}{4\pi k^2}, \quad (3.45)$$

where $E(k, 0)$ is the energy spectrum. It follows that if we adopt spherical polar coordinates and write

$$R(r-r_o, t, t|r_o, 0) = R(r, \theta, \phi, t),$$

(3.40) has the form

$$\frac{\partial R}{\partial t} = \frac{1}{r^2} \frac{\partial}{\partial r} \left[r^2 K(r, t) \frac{\partial R}{\partial r} \right] + \frac{1}{r^2} M(r, t) \left[\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial R}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 R}{\partial \phi^2} \right], \quad (3.46)$$

where, by (3.44) and (3.45)

$$K(r,t) = 4(2\pi)^3 \int_0^t dt' \int_0^\infty dk \left[\frac{1}{3} - \frac{\sin kr}{(kr)^3} + \frac{\cos kr}{(kr)^2} \right] E(k, t') \tilde{G}(k, t'), \quad (3.47)$$

$$M(r,t) = 2(2\pi)^3 \int_0^t dt' \int_0^\infty dk \left[\frac{2}{3} - \frac{\sin kr}{kr} + \frac{\sin kr}{(kr)^3} - \frac{\cos kr}{(kr)^2} \right] E(k, t') \tilde{G}(k, t'). \quad (3.48)$$

For small kr , the quantities in the square brackets $\left[\dots \dots \right]$ in the integrands of (3.47) and (3.48) are proportional to $(kr)^2$. Also, in the inertial range, we may take (cf. eq. 2.21)

$$\tilde{G}(k, t) = \frac{1}{(2\pi)^3} \exp \left[-\frac{1}{2} k^2 v_1^2 t^2 \right], \quad (3.49)$$

and (cf. Kraichnan, 1959, Sec. 8.4)

$$E(k, t) = E(k) \exp \left[-\frac{1}{2} k^2 v_1^2 t^2 \right]. \quad (3.50)$$

Thus, since by our initial supposition (3.35), $k^2 E(k) \rightarrow 0$, $k \rightarrow 0$, it follows that the integrands of (3.47) and (3.48) do not tend to infinity as rapidly as k^{-1} , $k \rightarrow 0$. Consequently, the form of $E(k)$ for small $k (\sim k_0)$ does not influence the values of $K(r, t)$ and $M(r, t)$ appreciably. However, these quantities do depend implicitly on the energy containing range through the forms (3.49) and (3.50) for $\tilde{G}(k, t)$ and $E(k, t)$. We will discuss this in more detail at the conclusion of this section.

Three further approximations are clearly justified. First since $\langle (r-r_o)^2 \rangle$ is large compared to r_o^2 for the times under consideration, r_o may be neglected in (3.47) and (3.48). Clearly, also, the particles have in this time lost all memory of the initial orientation of r , so that $R(r, \theta, \phi, t)$ is spherically symmetrical and, by (3.46), satisfies

$$\frac{\partial R(r, t)}{\partial t} = \frac{1}{r^2} \frac{\partial}{\partial r} \left[r^2 K(r, t) \frac{\partial R(r, t)}{\partial r} \right]. \quad (3.51)$$

Lastly, since $\langle r^2 \rangle$ is small compared to $(v_o t)^2$, the exponential factors of (3.49) and (3.50) are small at the upper limit of integration over t in (3.47) and (3.48), and we may therefore write

$$\begin{aligned} K(r, t) &= K(r) = 4 \int_0^\infty dt \int_0^\infty dk \left[\frac{1}{3} - \frac{\sin kr}{(kr)^3} + \frac{\cos kr}{(kr)^2} \right] E(k) \exp \left[-\frac{1}{3} k^2 v_o^2 t^2 \right] \\ &= 2\sqrt{3\pi} \int_0^\infty \frac{dk}{v_o k} \left[\frac{1}{3} - \frac{\sin kr}{(kr)^3} + \frac{\cos kr}{(kr)^2} \right] E(k). \end{aligned} \quad (3.52)$$

Equation (3.51) was proposed by Richardson (1926) for the intermediate times discussed here. To investigate further the form of the variable diffusion coefficient (3.52), we will assume that in the inertial range the spectrum is a power law * of the form (cf. eq. (3.35))

*

We may now take the wave number k_d characteristic of the dissipation range to be

$$k_d = R_o^{1/(3-n)} k_o,$$

(This footnote continued on p. 39)

$$E(k) = \beta \epsilon^{n-1} v_o^{5-3n} k^{-n}, \quad (1 < n < 2), \quad (3.53)$$

where β is a dimensionless constant of the order of unity. With this spectrum, (3.52) gives on integrating by parts and use of results of Watson (1944, Secs. 3.4, 13.24),

$$\begin{aligned} K(r) &= \frac{\beta \pi \sqrt{6}}{n} \epsilon^{n-1} v_o^{4-3n} r^n \int_0^\infty \frac{J_{5/2}(x) dx}{x^{n+3/2}} \\ &= \frac{\beta \pi \sqrt{3}}{2^{n+1} n} \frac{\Gamma(1 - \frac{1}{2}n)}{\Gamma(\frac{5}{2} + \frac{1}{2}n)} \epsilon^{n-1} v_o^{4-3n} r^n \equiv \lambda r^n, \quad (\text{say}). \end{aligned} \quad (3.54)$$

(Continuation of footnote on page 38.)

where

$$k_o \equiv 1/\ell = \epsilon/v_o^3, \quad R_o = v_o^4/\epsilon \nu = v_o/k_o \nu.$$

The "intermediate times" referred to above may now be defined more precisely by

$$T(r_o) = \left(\frac{r_o}{\ell} \right)^{(3-n)/2} \quad \frac{\ell}{v_o} \ll t \ll \frac{\ell}{v_o}.$$

Also, the circumstances under which (3.38) is a good approximation are

$$v_o \tau/\ell \gg (v_o t/\ell)^{1/(2-n)}.$$

Note that (3.53) may be written

$$E(k) = \beta \ell v_o^2 (k \ell)^{-n}.$$

If we suppose that at some time $t_1 \gg T(r_0)$

$$R(r, t) = \tilde{R}(r, t_1), \quad (3.55)$$

then at subsequent times t ($\ll \ell/v_0$), (3.51) and (3.54) show that

$$R(r, t) = \frac{1}{2}(2-n)r^{-\frac{1}{2}(n+1)} \int_0^\infty \tilde{R}(\xi, t_1) J_s \left(\xi r^{\frac{1}{2}(2-n)} \right) e^{-\frac{1}{4}\lambda(2-n)^2 \xi^2 (t-t_1)} \xi d\xi, \quad (3.56)$$

where $s = (1+n)/(2-n)$ and

$$\tilde{R}(\xi, t_1) = \int_0^\infty r^{\frac{1}{2}(3-n)} R(r, t) J_s \left(\xi r^{\frac{1}{2}(2-n)} \right) dr. \quad (3.57)$$

For $t \gg t_1$, (3.56) reduces to *

$$R(r, t) = \frac{1}{(2-n)^{4+n/2-n} \Gamma(\frac{3}{2-n})} \cdot \frac{1}{(\lambda t)^{\frac{3}{2-n}}} e^{-r^{2-n}/(2-n)^2 \lambda t}, \quad (3.58)$$

so that

$$\langle r^2 \rangle = (2-n)^{4/2-n} \frac{\Gamma(\frac{5}{2-n})}{\Gamma(\frac{3}{2-n})} (\lambda t)^{2/2-n}. \quad (3.59)$$

* The behavior of $R(r, t)$ and $\langle r^2 \rangle$ given by (3.58) and (3.59) is almost certainly independent of the approximation

$$R(r-r', t-t', t|r_0, 0) \doteq R(r, t, t|r_0, 0)$$

which led from (3.39) to (3.40). In fact, it can be shown that the more accurate approximation method based on (3.37) and (3.39) leads to results whose dimensional forms are identical to (3.58) and (3.59). However, it is worth confirming a posteriori that the approximation we have adopted is good.

The behavior of $\langle r^2 \rangle$ as a function of t is extraordinarily sensitive to the value of n assumed. Two cases are worthy of notice:

$n = 3/2$ (Kraichnan, 1959)

$$K(r) \propto \epsilon^{1/2} r^{3/2} / v_o^{1/2}, \quad \langle r^2 \rangle \propto \epsilon^{2/4} t^2 / v_o^2, \quad (3.60)$$

$n = 5/3$ (Kolmogorov, 1941)

$$K(r) \propto \epsilon^{2/3} r^{5/3} / v_o, \quad \langle r^2 \rangle \propto \epsilon^{4/6} t^6 / v_o^6. \quad (3.61)$$

Neither of these agree with the form proposed by Richardson (1926, see also Batchelor, 1950):

$$K(r) \propto \epsilon^{1/3} r^{4/3}, \quad \langle r^2 \rangle \propto \epsilon t^3. \quad (3.2)$$

That this is so is not surprising. Kraichnan's direct interaction approximation does not give an inertial range spectrum which agrees with that derived from Kolmogorov's similarity arguments. In the same way, when this approximation is applied to turbulent diffusion, it does not give a diffusion coef-

(Continuation of footnote on p. 40.)

For this purpose, we expand $R(\underline{r}-\underline{r}', t-t', t | \underline{r}_o, 0)$ in a series about $\underline{r}' = 0$, $t' = 0$; substitute in (3.39); and, using (3.58), compare the second term of the series with the first (cf. similar argument of Sec. 2.5). We find that this second term is smaller than the first by a factor of the order of $(r/v_o t)$ so that, for $t \ll l/v_o$ (at which times $\langle r^2 \rangle$ is small compared to l^2), we may expect (3.58) to be a good approximation to $R(r, t)$.

ficient which agrees with that derived by these similarity arguments, even if Kolmogorov's spectrum is assumed. This is because the dynamics of diffusion with a given velocity field differ on the two theories. It would seem that the behavior of $\langle r^2 \rangle$ as a function of t might provide a sensitive test by which to confront with experiment different assumptions about the structure of the inertial range.

Kraichnan (1959) has made a detailed comparison between the direct-interaction approximation and the Kolmogorov theory. He has traced the difference in the inertial range spectra to the difference in the role played by the energy-containing eddies in the two cases. In Kolmogorov's theory these eddies merely convect the small scale motions without influencing their dynamics, whereas in the direct-interaction approximation this is not so. In the same way, on arguments of the Kolmogorov type, the relative diffusion of particles should be independent of the energy-containing eddies. However, the application of the direct-interaction approximation has led to results which depend on the energy-containing motions. If we wish to modify our formalism in such a way that our results depend only on the small scale motions, we would transform to a frame of reference moving with the energy-containing eddies. We would expect that results of the form (3.40), (3.41) would be qualitatively correct, provided that $\tilde{G}(k, t)$ now described one-point diffusion relative to a source moving with the energy-containing eddies. On similarity arguments of the Kolmogorov type, $\tilde{G}(k, t)$ would then depend only upon $k^{2/3} \epsilon^{1/3} t$. Similarly $E(k, t)$ would describe the structure of the small-scale eddies in a frame of reference moving with the large scale motions, and would take the form

$$E(k, t) = E(k) f(k^{2/3} \epsilon^{1/3} t).$$

The overall effect of these modifications would be that $[kE(k)]^{1/2}$ would appear in place of v_o in the expression (3.52) for $K(r)$. Kraichnan (1959, Sec. 9.1) has shown that this substitution resolves the conflict between the Kolmogorov theory of turbulence and the theory based on the direct-interaction approximation. The quantity $[kE(k)]^{1/2}$ may be considered as the r.m.s. velocity associated with the motions of wave numbers k as they are convected by the large-scale motions. Substitution of this quantity for v_o in (3.52) gives

$$K(r) \propto \epsilon^{\frac{1}{2}(n-1)} v_o^{\frac{1}{2}(5-3n)} r^{\frac{1}{2}(n+1)}, \quad (3.63)$$

which, taking $n = 5/3$, leads to (3.62). With these changes in the interpretation of $\tilde{G}(k, t)$ and $E(k, t)$, the sensitivity of our results to the form of the inertial range spectrum remains.

Acknowledgement

I am extremely grateful to Dr. R.H. Kraichnan for detailed discussions and helpful criticisms of the work presented in this paper.

APPENDIX A:

Derivation of equation (2.26)

When the velocity field is spatially homogeneous, the problem of diffusion from a point source, although apparently possessing only radial symmetry even in the isotropic case, can always be rephrased as a homogeneous problem. For, since the equation (2.2) is linear, the response of the system to an initial disturbance

$$\psi(\underline{x}, t_0) = e^{i\underline{k} \cdot \underline{x}} , \quad (\text{A.1})$$

is

$$\langle \psi(\underline{x}, t) \rangle = \int d\underline{x}_0 G(\underline{x}, t | \underline{x}_0, t_0) e^{i\underline{k} \cdot \underline{x}_0} , \quad (\text{A.2})$$

and, since for a homogeneous velocity field, $G(\underline{x}, t | \underline{x}_0, t_0)$ depends on \underline{x} and \underline{x}_0 in the combination $\underline{x} - \underline{x}_0$ only, equation (A.2) can be rewritten

$$\langle \psi(\underline{x}, t) \rangle = (2\pi)^3 \tilde{G}(\underline{k}, t | t_0) e^{i\underline{k} \cdot \underline{x}} , \quad (\text{A.3})$$

where

$$\tilde{G}(\underline{k}, t | t_0) = \frac{1}{(2\pi)^3} \int G(\underline{x} - \underline{x}_0, t | t_0) e^{-i\underline{k} \cdot (\underline{x} - \underline{x}_0)} d(\underline{x} - \underline{x}_0) \quad (\text{A.4})$$

is the Fourier transform of the Green's function $G(\underline{x}, t | \underline{x}_0, t_0)$. Equation (A.3) proves the average response matrix of the Fourier modes is diagonal (when the velocity field is homogeneous) and that $(2\pi)^3 \tilde{G}(\underline{k}, t | t_0)$ is the average response function for mode \underline{k} .

Having established this correspondence, we will now derive an approximate equation for the response function by a method parallel to that

employed by Kraichnan (1959)* for the velocity field response function. For simplicity, we will suppose henceforth that the velocity field is also statistically stationary. The modifications necessary if the field is not stationary are easily included.

It is convenient to introduce the artifice of cyclic boundary conditions over a large cube of side L in order to expand $\psi(\underline{x}, t)$ and $\underline{u}(\underline{x}, t)$ in Fourier sums rather than Fourier integrals:

$$\psi(\underline{x}, t) = \sum_{\underline{k}} \tilde{\psi}(\underline{k}, t) e^{i\underline{k} \cdot \underline{x}}, \quad [\tilde{\psi}(\underline{k}) = \tilde{\psi}^*(-\underline{k})] \quad , \quad (\text{A.5})$$

$$\underline{u}(\underline{x}, t) = \sum_{\underline{k}} \tilde{u}(\underline{k}, t) e^{i\underline{k} \cdot \underline{x}}, \quad [\tilde{u}(\underline{k}) = \tilde{u}^*(-\underline{k})] \quad , \quad (\text{A.6})$$

(cf. K., eq. 2.1). Equation (2.2) may be written

$$\frac{\partial \tilde{\psi}(\underline{k}, t)}{\partial t} + ik_j \sum_{\underline{q}} \tilde{u}_j(\underline{p}, t) \tilde{\psi}(\underline{q}, t) = 0, \quad (\text{A.7})$$

where $\underline{p} = \underline{k} - \underline{q}$. The response function $\tilde{g}(\underline{k}, t)$ for mode \underline{k} is the solution of (A.7) under the initial conditions

$$\left. \begin{array}{l} \tilde{\psi}(\underline{k}, t) \equiv \tilde{g}(\underline{k}, t) = 1 \\ \tilde{\psi}(\underline{q}, t) \equiv \tilde{\psi}_{\underline{k}}(\underline{q}, t) = 0, \quad \underline{q} \neq \underline{k} \end{array} \right\} t = 0, \quad (\text{A.8})$$

* This paper will be designated by 'K' in these Appendices. Some differences in notation should be noted: In K., $g(\underline{k}, t)$ refers to the velocity field (impulse) response function and not to the response function for (A.9) below. Also, in K., g refers to an averaged response while, in this appendix, it does not; the average being denoted by $\langle g \rangle$. Further, the notational distinction between a quantity and its Fourier transform is different from that adopted in this paper.

(cf. K. Sec. 2.1). By the equation of motion $\tilde{\Psi}_{\underline{k}}(\underline{q}, t)$ and the direct interaction approximation, we find (cf. K. eq. 2.24) that for solution (A.8)

$$\tilde{\Psi}_{\underline{k}}(\underline{q}, t) = -iq_{\underline{k}} \int_0^t \tilde{u}_{\underline{k}}(-\underline{p}, t') \tilde{g}(\underline{k}, t') \tilde{g}(\underline{q}, t-t') dt' , \quad (\text{A.9})$$

Thus by (2.30) we have

$$\frac{\partial \tilde{g}(\underline{k}, t)}{\partial t} = - \sum_{\underline{q}} k_i k_j \int_0^t \tilde{u}_i(\underline{p}, t) \tilde{u}_j(-\underline{p}, t') \tilde{g}(\underline{q}, t-t') \tilde{g}(\underline{k}, t') dt' , \quad (\text{A.10})$$

and, on averaging, using the principle of weak statistical dependence (cf. K. Sec. 2.2 and eq. (2.25)), we find

$$\begin{aligned} \frac{\partial}{\partial t} \langle \tilde{g}(\underline{k}, t) \rangle &= - \sum_{\underline{q}} k_i k_j \int_0^t \left\langle \tilde{u}_i(\underline{p}, t) \tilde{u}_j(-\underline{p}, t') \right\rangle \\ &\quad \left\langle \tilde{g}(\underline{q}, t-t') \times \tilde{g}(\underline{k}, t') \right\rangle dt' . \end{aligned} \quad (\text{A.11})$$

Now let us take the limit. Make the transition $L \rightarrow \infty$. Let

$$\tilde{U}_{ij}(\underline{k}, t-t') = \lim_{L \rightarrow \infty} \left(\frac{L}{2\pi} \right)^3 \left\langle \tilde{u}_i(\underline{k}, t) \tilde{u}_j(-\underline{k}, t') \right\rangle , \quad (\text{A.12})$$

so that

$$U_{ij}(\underline{x}-\underline{x}', t-t') = \left\langle u_i(\underline{x}, t) u_j(\underline{x}', t') \right\rangle = \int \tilde{U}_{ij}(\underline{k}, t-t') e^{i\underline{k} \cdot (\underline{x}-\underline{x}')} d\underline{k} , \quad (\text{A.13})$$

and let

$$\tilde{G}(\underline{k}, t) = \lim_{L \rightarrow \infty} \left(\frac{L}{2\pi} \right)^3 \langle \tilde{g}(\underline{k}, t) \rangle \quad (\text{A.14})$$

so that equations (A.4, 5, 8) are consistent (cf. eqs. 3.2, 3.3). Then

$$\frac{\partial \tilde{G}(\underline{k}, t)}{\partial t} = -(2\pi)^3 k_i k_j \int_0^t dt' \int d\underline{q} \tilde{U}_{ij}(\underline{p}, t') \tilde{G}(\underline{q}, t') \tilde{G}(\underline{k}, t-t') . \quad (\text{A.15})$$

This result can be returned to physical space by writing it as

$$\frac{\partial \tilde{G}(\underline{k}, t)}{\partial t} = -k_i k_j \int_0^t dt' \int dp \int dq \int dx' \tilde{U}_{ij}(p, t') \tilde{G}(q, t') \tilde{G}(k, t-t') e^{ix' \cdot (p+q-k)} ; \quad (A.16)$$

i.e. (cf. A.4, 13)

$$\frac{\partial \tilde{G}(\underline{k}, t)}{\partial t} = -k_i k_j \int_0^t dt' \int dx' U_{ij}(x', t') G(x', t') \tilde{G}(k, t-t') e^{-ik \cdot x'} . \quad (A.17)$$

Using (A.4) again, this can be rewritten

$$\frac{\partial G(x, t)}{\partial t} = \frac{\partial^2}{\partial x_i \partial x_j} \int_0^t dt' \int dx' U_{ij}(x', t') G(x', t') G(x-x', t-t') . \quad (A.18)$$

We now give an alternative derivation of (A.15) based on summing selected terms from all orders of the formal expansion. For simplicity we will treat the one dimensional case only. Then (2.2) is replaced by

$$\frac{\partial \Psi(x, t)}{\partial t} = - \frac{\partial}{\partial x} [u(x, t) \Psi(x, t)] . \quad (A.19)$$

On making expansions analogous to (A.5, 6), we obtain from (A.19)

$$\frac{\partial \tilde{\Psi}(\underline{k}, t)}{\partial t} = -ik \sum_q \tilde{u}(p, t) \tilde{\Psi}(q, t) , \quad (A.20)$$

where $p=k-q$. (We will again suppose the mean flow $\tilde{u}(0, t)$ is zero.) The formal solution for the response function for mode k (cf. A.8) is

$$\begin{aligned}
\tilde{g}(k, t) = & 1 - \sum_{\ell+m=0} k(k-\ell) [\tilde{l} * \tilde{u}(\ell) * \tilde{u}(m)]_t \\
& + i \sum_{\ell+m+n=0} k(k-\ell)(k-\ell-m) [\tilde{l} * \tilde{u}(\ell) * \tilde{u}(m) * \tilde{u}(n)]_t \quad (A.21) \\
& + \sum_{\ell+m+n+r=0} k(k-\ell)(k-\ell-m)(k-\ell-m-n) [\tilde{l} * \tilde{u}(\ell) * \tilde{u}(m) * \tilde{u}(n) * \tilde{u}(r)]_t \dots,
\end{aligned}$$

where the $*$ denotes the process of convolution: -

$$[\tilde{f} * g]_t = \int_0^t f(t') g(t-t') dt' .$$

To determine the response $\tilde{\psi}_k(q, t)$ induced in mode q by the initial excitation in mode k , we can write (A.20) in the form

$$\begin{aligned}
\tilde{\psi}_k(q, t) + iq \int_0^t dt' \tilde{u}(-p, t') \tilde{g}(k, t') = & -iq \sum_{\ell \neq -p} \int_0^t \tilde{u}(\ell, t') \tilde{\psi}_k(q-\ell, t') dt', \\
(A.22)
\end{aligned}$$

and iterate starting with a zeroth approximation derived by setting $\tilde{\psi}_k = 0$ on the right, followed by a first approximation obtained by substituting the zeroth approximation for $\tilde{\psi}_k$ into the right-hand side of (A.22), and so on. In this way, we find

$$\begin{aligned}
\tilde{u}(p, t) \tilde{\psi}_k(q, t) = & -iqu(p, t) [\tilde{l} * \{ \tilde{u}(-p) \tilde{g}(k) \}]_t \\
& -qu(p, t) \sum_{\ell} (q-\ell) [\tilde{l} * \tilde{u}(\ell) * \{ \tilde{u}(-p-\ell) \tilde{g}(k) \}]_t \quad (A.23) \\
& +iqu(p, t) \sum_{\ell, m} (q-\ell)(q-\ell-m) [\tilde{l} * \tilde{u}(\ell) * \tilde{u}(m) * \{ \tilde{u}(-p-\ell-m) \tilde{g}(k) \}]_t \\
& \dots
\end{aligned}$$

where the sums exclude indices for which ℓ , $\ell + m$, $\ell + m + n$, are equal to $-p$.

So far, the analysis is exact. Now we make a selection of terms. Equation (A.7) shows that to each interaction which, through the agency of $\tilde{u}(p)$, induces a rate of change in $\tilde{\psi}(k)$ proportional to $\tilde{\psi}(k-p)$, there is a conjugate interaction which, through the agency of $\tilde{u}(-p)$, induces a rate of change in $\tilde{\psi}(k-p)$ which is proportional to $\tilde{\psi}(k)$. There are, of course, less direct couplings between $\tilde{\psi}(k)$ and $\tilde{\psi}(k-p)$ which involve more than one intermediate \tilde{u} mode, but, in the formal solution (A.21) for $\tilde{g}(k,t)$ and the expression (A.23) for $\tilde{u}(p,t) \tilde{\psi}_k(q,t)$, we will only retain a term if all the interactions of which it is composed occur in conjugate pairs. (We note that this automatically excludes all terms with an odd number of u factors.)

We can also explain our selection rule by a diagrammatic representation of the terms composing the series (A.21). A typical term containing $2n$ u -factors will be said to be of order n . The wave numbers of the $\tilde{\psi}(k-p)$ modes involved in the interaction with $\tilde{\psi}(k)$ form the segments of the base line of the diagram (see figure 1) and the ingoing arrow on the left marked k represents the mode $\tilde{\psi}(k)$ itself. The next segment, marked $(k-\ell)$, represents the $\tilde{\psi}(k-\ell)$ mode with which it interacts, the $\tilde{u}(\ell)$ mode concerned in the interaction being represented by the loop marked ℓ leaving the base line. The following intersections of loop and line have a similar meaning. The selection rule ensures that to every vertex A involving $\tilde{\psi}(s)$, $\tilde{u}(\ell)$, $\tilde{\psi}(s-\ell)$ there is a conjugate vertex A' involving $\tilde{\psi}(s-\ell)$, $\tilde{u}(-\ell)$, $\tilde{\psi}(s)$. Thus the sum of all the \tilde{u} -factors involved at vertices between A and A' is zero; i.e. there is an even number of such vertices and these

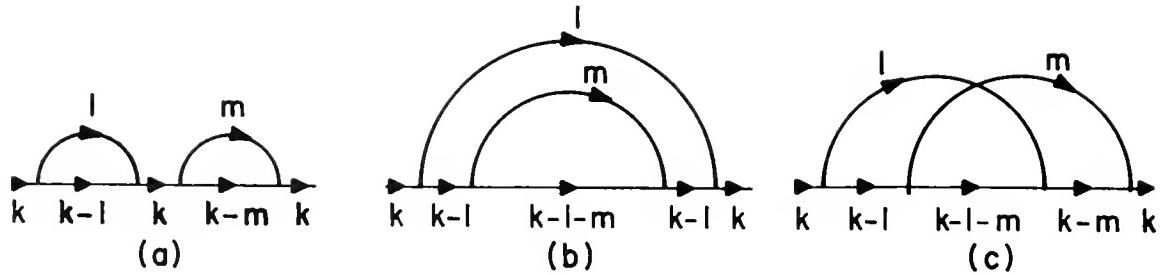


Fig. 1: Diagrams for $n = 2$ representing three non-zero terms of $\left[\bar{1} * u(\ell) * u(m) * u(n) * u(r) \right]_t$ of equation (A.21). (a) $\ell + m = n + r = 0$,
 (b) $\ell + r = m + n = 0$, (c) $\ell + n = m + r = 0$.

are conjugate in pairs. Thus the lines joining conjugate vertices do not cut each other.

Two consequences of the selection rule should be noted. First, the only terms of the formal expansion which are included are those which involve pairs $\tilde{u}(\ell)$, $\tilde{u}(-\ell)$. These terms are the same in normal and non-normal distributions having the same covariance. Secondly, not all terms involving pairs $\tilde{u}(\ell)$, $\tilde{u}(-\ell)$ are included; for they will not be conjugate vertices if the loop joining them cuts the loop joining another pair. For example in figure 1 all the interactions are shown for $n=2$ and, by the selection rule, we retain only l(a) and l(b).

Consider now a general term in the series (A.23).

$$iq(q-\ell)(q-\ell-m)\dots\tilde{u}(p,t)[\tilde{l}\ast\tilde{u}(\ell)\ast\tilde{u}(m)\ast\tilde{u}(n)\ast\dots\ast\{\tilde{u}(-\tilde{p}-\ell-m-n-\dots)\tilde{g}(k)\}]_t.$$

The initial $\tilde{u}(p,t)$ term must find a mate $\tilde{u}(-p)$ within the convolution product. Suppose it is not $\tilde{u}(-p-\ell-m-\dots)$. It cannot be any of the even placed terms $\tilde{u}(m)\dots$, for, if it were, there would be an odd number of vertices on the diagram between $\tilde{u}(p)$ and $\tilde{u}(-p)$ and these could not all find mates without cutting the loop joining $\tilde{u}(p)$ and $\tilde{u}(-p)$. The mate for $\tilde{u}(p)$ must therefore be one of the odd placed terms $\tilde{u}(\ell)$, $\tilde{u}(n)\dots$. There is then an even number of vertices on the diagram between $u(p)$ and $u(-p)$ and these vertices can and, by the selection rule, must be joined without cutting the loop joining $\tilde{u}(p)$ and $\tilde{u}(-p)$; i.e. the sum of the wave numbers $\ell + m + \dots$ for vertices lying between $\tilde{u}(p)$ and $\tilde{u}(-p)$ must be zero; in other words, the sum of these intermediate wave numbers together with the wave number $-p$ for the mate of p itself must equal $-p$. But these

terms are precisely the ones excluded in the summations of (A.26). Hence we have reached a contradiction. Thus, the mate for $\tilde{u}(p)$ must be $\tilde{u}(-p-\ell-m\dots)$ and the sum $\ell+m+\dots$ of all intermediate wave-numbers must be zero. Now compare the general term of the series (A.23) for $\tilde{u}(p,t) \tilde{\psi}_k(q,t)$ with the corresponding term [as derived from (A.21)] for the series for

$$-iq \tilde{u}(p,t) [\{ \tilde{u}(-p) \tilde{g}(k) \} * \tilde{g}(q)]_t .$$

It will be readily seen from the above argument that, under the selection rule, the two series are identical; i.e.

$$\tilde{u}(p,t) \tilde{\psi}_k(q,t) = -iq \int_0^t \tilde{u}(p,t) \tilde{u}(-p,t') \tilde{g}(k,t') \tilde{g}(q,t-t') dt' .$$

Hence, by (A.20), and the principle of weak statistical dependence (K. Sec. 2.2),

$$\frac{\partial \langle \tilde{g}(k,t) \rangle}{\partial t} = -k \sum_q q \int_0^t dt' \langle \tilde{u}(p,t) \tilde{u}(-p,t') \rangle \langle \tilde{g}(k,t') \rangle \langle \tilde{g}(q,t-t') \rangle , \quad (A.24)$$

as before. (This proof generalizes easily to 3-dimensions.)

As further confirmation, we can show, by the direct counting of diagrams, that for large $k (\gg \ell_0^{-1})$ the selection criterion gives

$$\langle \tilde{g}(k,t) \rangle = J_1(2kv_1 t) / kv_1 t , \quad (A.25)$$

which (cf. eq. 2.37) is also the solution to (A.24) under these circumstances. For large k , all the terms $k-\ell$, $k-\ell-m$, etc. of (A.21) are equal to k with negligible error. Also, any convolution term of order n is asymptotically equal to

$$(v_L t)^{2n} / (n)!$$

(since, as we can show a posteriori, $\langle \tilde{g}(k,t) \rangle$ is negligibly small unless $t \ll \ell/v_\infty$). Hence

$$\langle \tilde{g}(k,t) \rangle = \sum_0^\infty (-1)^n N_n (kv_L t)^{2n} / (2n)! \quad (\text{A.26})$$

where N_n is the number of non-intersecting diagrams of order n .

To determine N_n , we introduce the idea of a block. A diagram will be said to be composed of s blocks if there exist s particular conjugate pairs $A_1 A_1'$, $A_2 A_2'$, ..., $A_s A_s'$ with the following property: that both vertices of every other conjugate pair $B B'$ lie between $A_r A_r'$ for some choice of r . In other words, there are no loops in the diagram which (topologically) lie above any of the pairs $A_r A_r'$. For examples, see figure 2, in which the s loops composing the outline of the blocks are identified by heavy lines. Suppose the number of diagrams of order n composed of s blocks is $N_{n,s}$ ($1 \leq s \leq n$). We proceed by induction. We consider an s block diagram of order $n-1$, and add an additional loop whose left-hand vertex L lies to the left of the entire s block diagram. There are $(s+1)$ possible positions for the right-hand vertex R : P_1, P_2, \dots, P_{s+1} (see figure 3), and, if we choose P_r , the diagram of order n thereby generated has r blocks. Remembering that L is always to the left of the entire diagram of order $(n-1)$, it is not difficult to see that every diagram of order n can be generated without duplication by applying this process to every $(n-1)$ order diagram. Since an $(n-1)$ order s block diagram produces one diagram each of p blocks ($p=1, 2, \dots, s+1$), we see that

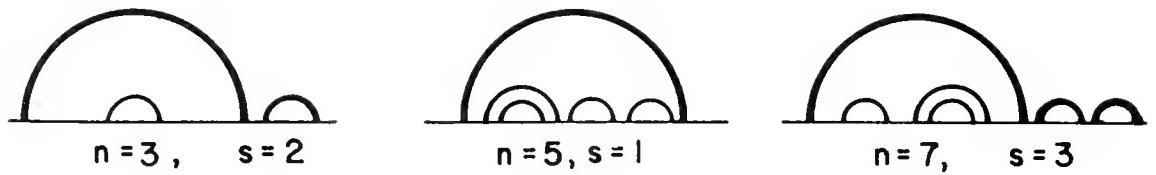


Fig. 2: Illustrations of block structure of diagrams.

Outlines of blocks are identified by heavy lines.

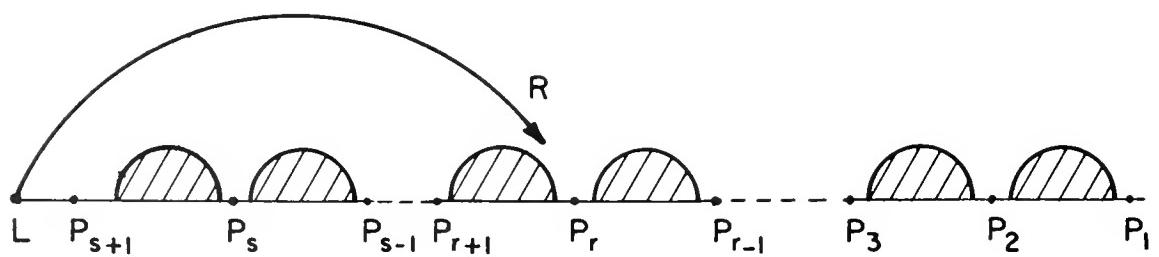


Fig. 3: The generation of a diagram of order n from a diagram of order $n - 1$. Shaded areas denote blocks.

$$N_{n,s} = \sum_{r=s-1}^{n-1} N_{n-1,r} , \quad (A.27)$$

with the convention $N_{n,0} = 0$. It follows that

$$N_{n,r} = \frac{r(2n-r-1)!}{n!(n-r)!} , \quad (A.28)$$

and, since every n order diagram gives rise to one, and only one, $n+l$ order, 1 block diagram, we have

$$N_n = N_{n+l,1} = \frac{(2n)!}{n!(n+l)!} . \quad (A.29)$$

Therefore, by (A.26) ,

$$\langle \tilde{g}(k,t) \rangle = \sum_0^{\infty} (-1)^n \frac{(kv_1 t)^{2n}}{n!(n+l)!} = \frac{J_l(2kv_1 t)}{kv_1 t} . \quad (A.30)$$

APPENDIX B:

Derivation of equation (3.18)

As in appendix A, we will suppose that the velocity field is homogeneous and stationary, and consider homogeneous forms of ψ_1 and ψ_2 . We again adopt cyclic boundary conditions and expansions of the form (A.5, 6);

$$\psi_1(\underline{x}, t) = \sum_{\underline{k}} \tilde{\psi}_1(\underline{k}, t) e^{i \underline{k} \cdot (\underline{x} - \underline{x}_0)}, \quad (B.1)$$

$$\psi_2(\underline{y}, s) = \sum_{\underline{k}} \tilde{\psi}_2(\underline{k}, s) e^{i \underline{k} \cdot (\underline{y} - \underline{y}_0)}, \quad (B.2)$$

$$\underline{u}(\underline{x}, t) = \sum_{\underline{k}} \tilde{u}(\underline{k}, t) e^{i \underline{k} \cdot \underline{x}}, \quad (B.3)$$

(cf. K. eq. 2.1 and footnote to Appendix A). Then equation (2.2) for the ψ_1 field may be written

$$\frac{\partial \tilde{\psi}_1(\underline{k}, t)}{\partial t} + i k_i \sum_{\underline{q}} \tilde{u}_i(\underline{p}, t) e^{-i \underline{p} \cdot \underline{x}_0} \tilde{\psi}_1(\underline{q}, t) = 0, \quad (B.4)$$

and, for the ψ_2 field,

$$\frac{\partial \tilde{\psi}_2(\underline{k}, s)}{\partial s} + i k_i \sum_{\underline{q}} \tilde{u}_i(\underline{p}, s) e^{-i \underline{p} \cdot \underline{y}_0} \tilde{\psi}_2(\underline{q}, s) = 0, \quad (B.5)$$

where $\underline{p} = \underline{k} - \underline{q}$. By (B.4) we have (cf. K. eqs. 3.4, 3.5)

$$\frac{\partial}{\partial t} \langle \tilde{\psi}_1(-\underline{k}, t) \tilde{\psi}_2(\underline{k}, s) \rangle = i k_i \sum_{\underline{q}} \langle \tilde{u}_i(-\underline{p}, t) \tilde{\psi}_1(-\underline{q}, t) \tilde{\psi}_2(\underline{k}, s) \rangle e^{i \underline{p} \cdot \underline{x}_0}. \quad (B.6)$$

Now, in the limit $L \rightarrow \infty$, the direct interaction approximation (K. Sec. 2.4) gives

$$\begin{aligned} & \langle \tilde{u}_i(\underline{p}, t) \tilde{\psi}_1(-\underline{q}, t) \tilde{\psi}_2(\underline{k}, s) \rangle \\ &= \langle \tilde{u}_i(\underline{p}, t) \tilde{\psi}_{1,-\underline{k}}(-\underline{q}, t) \tilde{\psi}_2(\underline{k}, s) \rangle + \langle \tilde{u}_i(\underline{p}, t) \tilde{\psi}_1(-\underline{q}, t) \tilde{\psi}_{2,\underline{q}}(\underline{k}, s) \rangle , \end{aligned} \quad (\text{B.7})$$

where (cf. eq. A.9 and K. Sec. 3)

$$\tilde{\psi}_{1,-\underline{k}}(-\underline{q}, t) = \int_{t_0}^t i q_j \tilde{u}_j(\underline{p}, t') \tilde{\psi}_1(-\underline{k}, t') \tilde{g}(\underline{q}, t-t') e^{-ip \cdot \underline{x}_0} dt' , \quad (\text{B.8})$$

$$\tilde{\psi}_{2,\underline{q}}(\underline{k}, s) = - \int_{s_0}^s i k_j \tilde{u}_j(\underline{p}, s') \tilde{\psi}_2(\underline{q}, s') \tilde{g}(\underline{k}, s-s') e^{-ip \cdot \underline{y}_0} ds' . \quad (\text{B.9})$$

Thus, applying the weak statistical dependence principle (K. Sec. 2.2), we find that, in the limit $L \rightarrow \infty$, (cf. eqs. A.12, 13, 14)

$$\begin{aligned} & \frac{\partial}{\partial t} \tilde{R}(\underline{k}, t, s | \underline{r}_0, \tau_0) = \\ & - (2\pi)^3 k_i k_j \int_0^t dt' \int d\underline{q} \tilde{U}_{ij}(\underline{p}, t') \tilde{G}(\underline{q}, t') \tilde{R}(\underline{k}, t-t', s | \underline{r}_0, \tau_0) \\ & + (2\pi)^3 k_i k_j \int_0^s ds' \int d\underline{q} \tilde{U}_{ij}(\underline{p}, \tau_0 - t+s-s') e^{ip \cdot \underline{r}_0} \tilde{G}(\underline{q}, s') \tilde{R}(\underline{k}, t, s-s' | \underline{r}_0, \tau_0) . \end{aligned} \quad (\text{B.10})$$

Returning to \underline{x} -space by the arguments of (A.15) to (A.18) this becomes

$$\frac{\partial}{\partial t} R(\underline{r}, t, s | \underline{r}_o, \tau_o) =$$

$$\frac{\partial^2}{\partial r_i \partial r_j} \left\{ \int_o^t dt' \int d\underline{r}' U_{ij}(\underline{r}', t') G(\underline{r}', t') R(\underline{r}-\underline{r}', t-t', s | \underline{r}_o, \tau_o) \right. \\ \left. - \int_o^s ds' \int d\underline{r}' U_{ij}(\underline{r}_o + \underline{r} - \underline{r}', \tau_o - t + s - s') G(\underline{r}', s') R(\underline{r}-\underline{r}', t, s-s' | \underline{r}_o, \tau_o) \right\} \quad (B.11)$$

which is equivalent to (3.18).

An alternative derivation of (B.11) can be given which is based on summing selected terms from all orders of the formal expansion of R . We will not discuss this method in detail, but we will describe how it differs from the similar method of appendix A, and for this purpose we will once more discuss the one dimensional case. The formal solution of (B.4) under the initial condition $\psi_1(x, t) = \delta(x - x_o)$ is given by

$$L^3 \tilde{\psi}_1(k, t) = l+i \sum_{\ell} k [l * \tilde{u}(\ell)]_t e^{-ilx_o} - \sum_{\ell, m} k(k+\ell) [l * \tilde{u}(\ell) * \tilde{u}(m)]_t e^{-i(\ell+m)x_o} \\ - i \sum_{\ell, m, n} k(k+\ell)(k+\ell+m) [l * \tilde{u}(\ell) * \tilde{u}(m) * \tilde{u}(n)]_t e^{-i(\ell+m+n)x_o} + \dots \quad (B.12)$$

Multiply this series term by term by the similar expansion for $L^3 \tilde{\psi}_2(k, t)$ and average, retaining only conjugate interactions. Now, however, the mate of $\tilde{u}(p)$ in one series may occur in a term of the other series, and it is precisely such interactions which, on the direct interaction approxi-

mation, give rise to the correlation between ψ_1 and ψ_2 . As in Appendix A, a diagrammatic representation may be made of the formal expansions. Now, however, there are two base lines, one for $\tilde{\psi}_1$ and one for $\tilde{\psi}_2$. The \tilde{u} modes are represented by loops joining conjugate vertices either on the same or on different base lines, and, as in Appendix A, these loops cannot cross. A typical term of the series for $\tilde{\psi}_1(-k)\tilde{\psi}_2(k)$ is shown diagrammatically in figure 4: it is obtained from the product of a second order term in $\tilde{\psi}_1(-k)$ and a fourth order term in $\tilde{\psi}_2(k)$.

On the basis of this selection rule of Appendix A, we will now recover the solution of Sec. 3.3 for large k . Under these circumstances, all the terms $k + \ell$, $k + \ell + m$, etc. of (B.12) are equal to k with negligible error. Also, a pair of conjugate vertices will, asymptotically, give a contribution of

$$\sum_{\ell} \tilde{u}(-\ell, o) \tilde{u}(\ell, o) = v_1^2 ,$$

if both vertices refer to $\tilde{\psi}_1$ or $\tilde{\psi}_2$, and give a contribution of

$$\sum_{\ell} \tilde{u}(-\ell, t_o) e^{i\ell x_o} \tilde{u}(\ell, s_o) e^{-i\ell y_o} = \langle u(x_o, t_o) u(y_o, s_o) \rangle \equiv U_{11}(r_o, \tau_o),$$

if one vertex refers to $\tilde{\psi}_1$ and the other to $\tilde{\psi}_2$. Hence

$$\langle \tilde{\psi}_1(k, t) \tilde{\psi}_2(k, s) \rangle =$$

$$\sum_{n, m, \ell} \frac{(-1)^{m+n}}{(\ell+2m)!(\ell+2n)!} \left[\frac{U_{11}(r_o, \tau_o)}{v_1^2} \right]^{\ell} N_{m; n; \ell} (kv_1 t)^{\ell+2m} (kv_1 s)^{\ell+2n}, \quad (B.13)$$

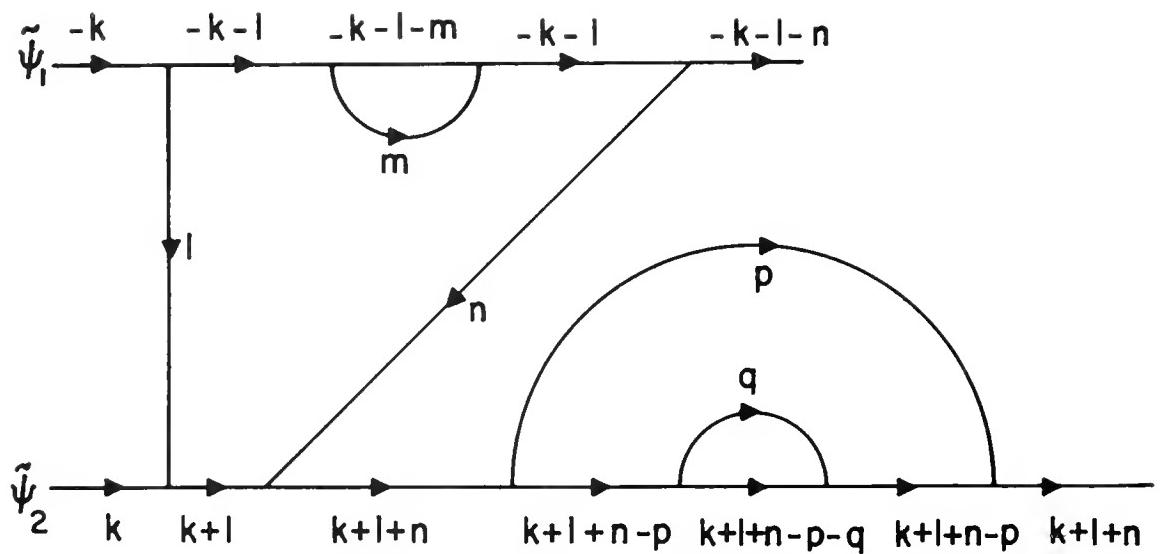


Fig. 4: A typical term in the formal expansion of $\langle \tilde{\psi}_1 \tilde{\psi}_2 \rangle$.

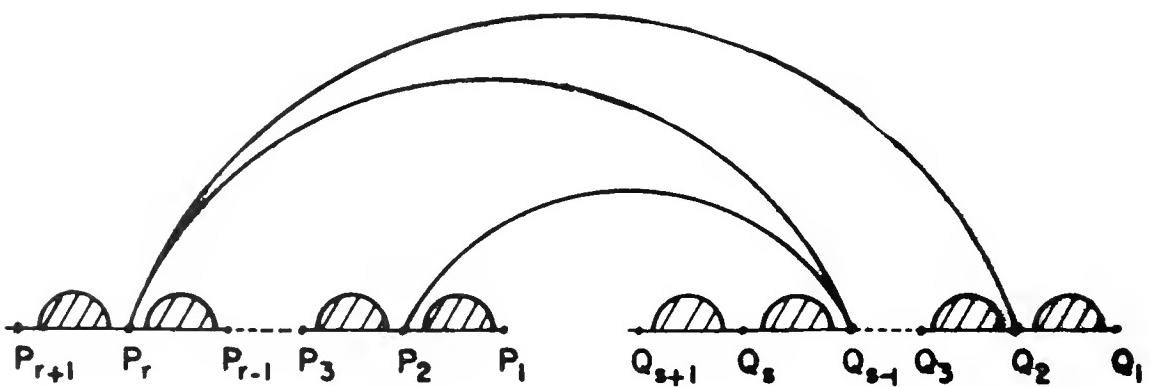


Fig. 5: A general term $\langle \tilde{\psi}_1 \tilde{\psi}_2 \rangle$ showing location of termini of links. Blocks are shaded.

where $N_{m;n;\ell}$ is the number of non-intersecting diagrams with $\ell + 2m$ points on the $\tilde{\Psi}_1$ base line, $\ell + 2n$ points on the $\tilde{\Psi}_2$ base line and ℓ links between them.

To determine $N_{m;n;\ell}$, we first evaluate the number $N(m,r; n,s; \ell)$ of diagrams in which the $\ell + 2m$ points lying on the $\tilde{\Psi}_1$ base line are composed of r blocks and ℓ links, and the $\ell + 2n$ points on the $\tilde{\Psi}_2$ base line are composed of s blocks and ℓ links. For example, figure 4 shows one diagram counted in $N(2,1; 4, 1; 2)$. There are $(r+1)$ possible locations P_1, P_2, \dots, P_{r+1} for the termini of the ℓ links on the $\tilde{\Psi}_1$ base line and $(s+1)$ possible locations Q_1, Q_2, \dots, Q_{s+1} for the termini on the $\tilde{\Psi}_2$ base line (see figure 5), and any number of termini can be placed at any of these locations. There are therefore $(r+\ell)! / r! \ell!$ ways of placing the ℓ termini on the $\tilde{\Psi}_1$ base line and $(s+\ell)! / s! \ell!$ ways of placing the other ℓ termini on the $\tilde{\Psi}_2$ base line. There is only one way of joining the ℓ termini on each base line with non-intersecting links. Hence, in all, there are

$$\frac{(r+\ell)!}{r!\ell!} \quad \frac{(s+\ell)!}{s!\ell!}$$

ways of linking the base lines. As we have seen (cf. A.28), there are

$$\frac{r(2m-r-1)!}{m!(m-r)!} \quad \frac{s(2n-s-1)!}{n!(n-s)!}$$

ways of forming the blocks on base lines. Hence.

$$N(m,r; n,s; \ell) = M(m,r; \ell) M(n,s; \ell),$$

where

$$M(m, r; \ell) = \frac{(r+\ell)!}{r! \ell!} \cdot \frac{r(2m-r-1)!}{m!(m-r)!} . \quad (B.14)$$

It follows that

$$N_{m;n;\ell} = \left[\sum_{r=1}^m M(m, r; \ell) \right] \left[\sum_{s=1}^n M(n, s; \ell) \right] . \quad (B.15)$$

Now, let

$$S_{m,\ell} = \ell! m! \sum_{r=1}^m M(m, r; \ell) = \sum_{\circ}^{m-1} \frac{(m+\ell-r)!}{r!} \frac{(m+r-1)!}{(m-r-1)!} . \quad (B.16)$$

From the calculations of Appendix A (cf. A.28, A.29), we know that

$$S_{m,-1} = (2m-1)! / m! , \quad S_{m,0} = (2m)! / (m+1)! , \quad (B.17)$$

and it is easy to establish from (B.16) that

$$S_{m,\ell} + S_{m+1,\ell-1} = (2m+\ell)S_{m,\ell-1} + \ell S_{m+1,\ell-2} . \quad (B.18)$$

By this recurrence relation, we may prove from (B.17) by induction that,

$$S_{m,\ell} = \frac{(\ell+1)!(\ell+2m)!}{m!(m+\ell+1)!} ; \quad (B.19)$$

i.e.

$$\sum_{r=1}^m M(m, r; \ell) = \frac{(\ell+1)(\ell+2m)!}{m!(m+\ell+1)!} . \quad (B.20)$$

Returning to (B.13) and (B.15) we find

$$\begin{aligned}
 & \tilde{\psi}_1(-k, t)\tilde{\psi}_2(k, s) \\
 = & \sum_{n, m, \ell} \frac{(-1)^{m+n}(\ell+1)^2}{m! n! (m+\ell+1)! (n+\ell+1)!} \left[\frac{U_{11}(r_o, \tau_o)}{v_1^2} \right]^\ell (kv_1 t)^{\ell+2m} (kv_1 s)^{\ell+2n} \\
 = & \sum_{\ell} (\ell+1)^2 \left[\frac{U_{11}(r_o, \tau_o)}{v_1^2} \right]^\ell \frac{J_{\ell+1}(2kv_1 t)}{kv_1 t} \frac{J_{\ell+1}(2kv_1 s)}{kv_1 s} ,
 \end{aligned} \tag{B.21}$$

which is equivalent to (3.31).

REFERENCES

- Batchelor, G.K., 1950. "The application of the similarity theory of turbulence to atmospheric diffusion". Quart. J. Roy. Met. Soc. 76, 133-146.
- Batchelor, G.K., 1952a. "Diffusion in a field of homogeneous turbulence II: The relative motion of particles". Proc. Camb. Phil. Soc. 48, 345-363.
- Batchelor, G.K., 1952b. "The effect of homogeneous turbulence on material lines and surfaces". Proc. Roy. Soc. Lond. A, 213, 349-366.
- Batchelor, G.K., 1953. "The theory of homogeneous turbulence". Cambridge: at the University Press.
- Corrsin, S., 1960. "Progress report on some turbulent diffusion research". Proc. Symposium on Atmospheric Diffusion and Air Pollution, held at Oxford, 1958, p. 161-164. Academic Press.
- Kolmogorov, A.N., 1941. "The local structure of turbulence in an incompressible viscous fluid for very large Reynolds numbers". C.R. Acad. Sci. U.R.S.S. 30, 301.
- Kraichnan, R.H., 1959. "The structure of isotropic turbulence at very high Reynolds numbers". J. Fluid Mech. 5, 497-543. This paper is referred to as 'K.' in the appendices.
- Kraichnan, R.H., 1960. "Dynamics of non-linear stochastic systems". New York University, Institute of Mathematical Sciences, Division of Electromagnetic Research, Report No. HT-7.
- Reid, W.H., 1955. "On the stretching of material lines and surfaces in isotropic turbulence with zero fourth cumulants". Proc. Camb. Phil. Soc. 51, 350-362.
- Richardson, L.F., 1926. "Atmospheric diffusion on a distance-neighbour graph". Proc. Roy. Soc. Lond. A, 110, 709-737.

- Roberts, P.H., 1957 "On the application of a statistical approximation to the theory of turbulent diffusion".
J. Rat. Mech. and Anal. 6, 781-800.
- Roberts, P.H. and Ursell, H.D., 1959 "Random walk on a sphere and on a Riemannian manifold".
Phil. Trans. Roy. Soc. Lond. A, 252, 317-356.
- Saffman, P.G. "An approximate calculation of the Lagrangian correlation coefficient for turbulent diffusion". Unpublished.
- Taylor, G.I., 1921 "Diffusion by continuous movements".
Proc. Lond. Math. Soc., 20, 196.
- Watson, G.N., 1944 "A treatise on the theory of Bessel functions".
2nd edition. Cambridge, at the University Press.

INITIAL DISTRIBUTION LIST FOR RESEARCH REPORTS

Contract No. N(onr)285-33, Task #P 062-206

(ONE copy unless otherwise noted)

Chief of Naval Research
Department of the Navy
Washington 25, D.C.

Attn: Codes 438
411
461
466
(3)

Commanding Officer
Office of Naval Research
Branch Office
495 Summer Street
Boston 10, Massachusetts
(0)

Commanding Office
Office of Naval Research
Branch Office
The John C. Crerar Library Bldg.
86 East Randolph Street
Chicago 1, Illinois

Commanding Officer
Office of Naval Research
Branch Office
346 Broadway
New York 13, New York
(2)

Commanding Officer
Office of Naval Research
Branch Office
1030 East Green Street
Pasadena 1, California

Commanding Officer
Office of Naval Research
Navy #100, Fleet Post Office
New York, New York
Attn: Mr. M.P. Tulin
(25)

Director
Naval Research Laboratory
Washington 25, D.C.
Attn: Code 2000 (Tech.
Info. Off.)
(6)

Chief, Bureau of Aeronautics
Department of the Navy
Washington 25, D.C.
Attn: Codes AD-3 (Aero & Hydro.Br.)
RS (Research Div.)
TD-42 (Library Sect.)

Chief, Bureau of Ordnance
Department of the Navy
Washington 25, D.C.
Attn: Codes ReO (Director, Ord. Sci.)
Ad3 (Tech. Lib. Br.)

Chief, Bureau of Ships
Department of the Navy
Washington 25, D.C.
Attn: Codes 106 (Tech. Asst. to Chief)
310 (Res. & Dev. Div.)
312 (Tech. Lib.)
442 (Sci. & Res.)
370 (Appl. Sci. Div.)
425 (Adv. Conc. Br.)

Commanding Officer and Director
David Taylor Model Basin
Washington 7, D.C.
Attn: Codes 142 (Library Br.)
500 (Tech. Dir. Hydromech. Lab.)
513 (Contract Res. Admin.)
550 (Underwater Acoustics Div.)

Chief, Bureau of Yards and Docks
Department of the Navy
Washington 25, D.C.
Attn: Code D-400 (Res. Div.)

Commander
Naval Ordnance Laboratory
White Oak, Maryland
Attn: Desk DA (Assoc. Tech. Dir. Aerobal.)
HL (Library Div.)

Commander
Naval Ordnance Test Station
China Lake, California
Attn: Code 753 (Library Div.)

Officer-in-Charge, Pasadena Annex
Naval Ordnance Test Station
3203 E. Foothill Blvd.
Pasadena 8, California
Attn: Codes P804 (Guid. & Control Div.)
P807 (Appl. Mech. Br.)
P8074 (Hydro. Br.)

Commanding Officer
U.S. Naval Underwater Ordnance Station
Newport, Rhode Island

Commanding Officer and Director
U.S. Naval Engineering Experiment Station
Annapolis, Maryland

Superintendent
U.S. Naval Postgraduate School
Monterey, California
Attn: Librarian

Superintendent
U.S. Naval Academy
Annapolis, Maryland
Attn: Librarian

Commanding Officer and Director
U.S. Navy Electronics Laboratory
San Diego 52, California
Attn: Mr. C. Miller

Commanding Officer and Director
U.S. Navy Underwater Sound Laboratory
Fort Trumbull, New London,
Connecticut
Attn: Mr. W.A. Von Winkle

Air Force Office of Scientific Research
Mechanics Division
Washington 25, D.C.

Commander
Air Force Cambridge Research Center
230 Albany Street
Cambridge 39, Massachusetts
Attn: Geophysical Research Library

Commanding Officer
Office of Ordnance Research
Box 61, Duke Station
Durham, North Carolina

Director of Research
National Advisory Committee for
Aeronautics
1512 H Street, N.W.
Washington 25, D.C.

Director
Langley Aeronautical Laboratory
National Advisory Committee for
Aeronautics
Langley Field, Virginia

Director
Moffett Field, California
National Advisory Committee for
Aeronautics

Director
Lewis Flight Propulsion Laboratory
National Advisory Committee for
Aeronautics
Cleveland, 11, Ohio

National Bureau of Standards
Washington 25, D.C.
Attn: Fluid Mechanics Section
Dr. G.B. Schubauer
Dr. P.S. Klebanoff
Aerodynamics Section
Dr. C.M. Tchen

Armed Services Technical
Information Agency
Document Service Center
Arlington Hall Station
Arlington 12, Virginia
(5)

Office of Technical Services
Department of Commerce
Washington 25, D.C.

Institute of Mathematical Sciences
New York University
25 Waverly Place
New York 3, New York
Attn: Prof. R. Courant
Dr. R. Kraichnan

Prof. W.J. Pierson, Jr.
Oceanography Department
New York University
University Heights
New York 53, New York

The John Hopkins University
Baltimore 18, Maryland
Attn: Mech. Engrg. Dept.
Prof. S. Corrsin
Prof. O.M. Phillips
Aero. Dept.
Prof. F.H. Clauser
Prof. L.S.G. Kovasznay
Civil Engrg. Dept.
Prof. R.K. Long

Institute for Fluid Dynamics and
Applied Mathematics
University of Maryland
College Park, Maryland
Attn: Prof. J.M. Burgers
Prof. F.R. Hama

Division of Applied Mathematics
Brown University
Providence 12, Rhode Island
Attn: Prof. M. Holt
Dr. R.E. Meyer
Dr. W.H. Reid

Harvard University
Cambridge 38, Massachusetts
Attn: Engrg. and Appl. Phys. Div.
Prof. H.W. Emmons
Math. Dept.
Prof. C. Birkhoff
Prof. S. Goldstein

Massachusetts Institute of Technology
Cambridge 39, Massachusetts
Attn: Math. Dept.
Prof. C.C. Lin
Hydro. Lab.
Prof. A.T. Ippen
Prof. J.W. Daily

University of Michigan
Ann Arbor, Michigan
Attn: Aero. Engrg. Dept.
Prof. A.M. Kuethe
Prof. M.S. Uberoi
Prof. W.W. Willmarth
Appl. Mech. Dept.
Prof. C.S. Yih
Civil Engrg. Dept.
Prof. V.L. Streeter
Nav. Arch. Dept.
Prof. R.B. Couch

Dr. J. C. Johnson
Engineering Research Institute
University of Michigan
Ann Arbor, Michigan

Iowa Institute of Hydraulic Research
State University of Iowa
Iowa City, Iowa
Attn: Prof. H. Rouse
Prof. L. Landweber

California Institute of Technology
Pasadena 4, California
Attn: Prof. C.B. Millikan
Prof. T.Y. Wu
Prof. H.W. Liepmann
Prof. D.S. Coles
Hydro. Lab.

University of California
Department of Engineering
Berkeley 4, California
Attn: Prof. H.A. Einstein
Prof. L. Grossman
Prof. J.V. Wehausen

Director
Experimental Towing Tank
Stevens Institute of Technology
711 Hudson Street
Hoboken, New Jersey

Director
Alden Hydraulic Laboratory
Worcester Polytechnic Institute
Worcester, Massachusetts

Director
Ordnance Research Laboratory
Pennsylvania State University
University Park, Pennsylvania

Dr. E. Skudryk
Ordnance Research Laboratory
Pennsylvania State University
University Park, Pennsylvania

Director
St. Anthony Falls Hydraulic Laboratory
University of Minnesota
Minneapolis 14, Minnesota

Director
Scripps Institution of
Oceanography
University of California
La Jolla, California

Director
Woods Hole Oceanographic
Institute
Woods Hole, Massachusetts

Director
Applied Physics Laboratory
The John Hopkins University
8621 Georgia Avenue
Silver Spring, Maryland

Dr. Th. von Karman
1051 S. Marengo Street
Pasadena, California

Prof. W. R. Sears, Director
Graduate School of Aeronautical
Engineering
Cornell University
Ithaca, New York

Dr. J. M. Robertson
Department of Theoretical and Applied
Mechanics
College of Engineering
University of Illinois
Urbana, Illinois

Prof. M. M. Munk, Head
Aeronautical Engineering Department
Catholic University of America
Washington, D.C.

Prof. A. Ferri, Head
Aeronautical Engineering and Applied
Mechanics Department
Polytechnic Institute of Brooklyn
99 Livingston Street
Brooklyn 1, New York

Profs. W.D. Hayes and L.L. Crocco
Department of Aeronautical Engineering
Princeton University
Princeton, New Jersey

Prof. G. Kuerti
Department of Mechanical Engineering
Case Institute of Technology
Cleveland, Ohio

Dr. F. K. Moore
Cornell Aeronautical Laboratory
Cornell Research Foundation
Box 235
Buffalo 21, New York

Prof. S. Chandrasekhar
Yerkes Observatory
Williams Bay, Wisconsin

Prof. H. Mac
Mathematics Department
Cornell University
Ithaca, New York

Dr. J. Laufer
Jet Propulsion Laboratory
California Institute of Technology
Pasadena, California

Prof. S. Eskinazi
Department of Mechanical Engineering
Syracuse University
Syracuse, New York

Dr. S. Kline
Department of Mechanical Engineering
Stanford University
Stanford, California

Institute of Aerophysics
University of Toronto
Toronto 5, Canada

Prof. J. W. Jackson
Engineering Experiment Station
Georgia Institute of Technology
Atlanta, Georgia

Prof. E. R. G. Eckert
Department of Mechanical Engineering
University of Minnesota
Minneapolis 14, Minnesota

Dr. E. P. Van Driest
Missile Development Division
North American Aviation, Inc.
Downey, California

Dr. I. Dyer
Bolt, Beranek and Newman, Inc.
50 Moulton Street
Cambridge, Massachusetts

Colorado State University
Fort Collins
Colorado
Attn: Prof. J.E. Cermak
Lept. of Civil Engineering

Hydronautics, Inc.
200 Monroe St.
Rockville, Md.
Attn: Mr. Phillip Eisenberg
Mr. Marshall P Tulin

University of Cincinnati
Cincinnati 31, Ohio
Attn: Library, Dept. of Physics

Western Electro-Acoustic Lab., Inc.
11789 San Vicente Blvd.
Los Angeles 49, California

This book may be kept

FOURTEEN DAYS

A fine will be charged for each day the book is kept overtime.

ILL NOAA Boulder Col ^{Unit 5-16-75}
6-6-75

GAYLORD 142

PRINTED IN U.S.A.

LIBRARY
UNIV-2

c.?

Robert^a
analytical theory of turbulent
diffusion

LIBRARY
UNIV-2

c.?

AUTHOR
"analytical theory of
turbulent diffusion"

DATE DUE

Borrower's Name

ROOM
NUMBER

Aut 16 '71

(LTC METZ, Standard 61, due 6/6/71)

**N. Y. U. Institute of
Mathematical Sciences**

25 Waverly Place
New York 3, N. Y.

NEW YORK
CITY LIBRARIES
251 Mercer St., New York, N.Y. 10012

